Algebraic K-Theory as a Motivic Space

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0. Introduction

Let M be a nice topological space, for example a manifold. For a natural number r, let us write $\mathrm{Vect}_r(M)$ for the set of isomorphism classes of real vector bundles over M of rank r. One of the most important properties of $\mathrm{Vect}_r(M)$ is its *homotopy invariance*: Pulling back along the projection map $M \times \mathbb{R} \to M$ induces an isomorphism

$$\mathrm{Vect}_r(M) \xrightarrow{\cong} \mathrm{Vect}_r(M \times \mathbb{R}).$$

In other words: Any rank r vector bundle over $M \times \mathbb{R}$ is isomorphic to the pullback of a rank r vector bundle over M.

In his famous paper [Ser55] from 1955 Serre asked, whether for a field k every finitely generated locally free module over the polynomial ring $k[X_1, ...X_n]$ is free. Defining an *algebraic vector bundle of rank r* over a scheme X to be a locally free O_X-module of constant rank r, this conjecture may be reformulated as follows: Pulling back along the projection map $\mathbb{A}^n \to \mathrm{Spec}\,(k)$ induces an isomorphism

$$* = \mathrm{Vect}_r(\mathrm{Spec}\,(k)) \to \mathrm{Vect}_r(\mathbb{A}^n).$$

Here we let $\mathrm{Vect}_r(X)$ denote the set of isomorphism classes of algebraic vector bundles of rank r over a scheme X. More than 20 years later, this conjecture was finally shown to be true by Quillen and Suslin in [Qui76] and [Sus76]. In fact even more was shown, namely that Serre's conjecture is even true when the field k is replaced by any Dedekind domain. This leads to the following conjecture, known as the Bass-Quillen conjecture:

Conjecture 0.1 (Bass-Quillen)**.** Let R be a regular noetherian ring of finite Krull dimension. Then the projection $\mathrm{Spec}\,(R) \times \mathbb{A}^1 \to \mathrm{Spec}\,(R)$ induces an isomorphism

$$\mathrm{Vect}_r(\mathrm{Spec}\,(R)) \to \mathrm{Vect}_r(\mathrm{Spec}\,(R) \times \mathbb{A}^1).$$

Today, this conjecture is known to be true in many special cases, but is still open in full generality. However, it strongly suggests that there should be some version of a homotopy theory of smooth schemes over a fixed base S, in which the affine line $\mathbb{A}^1_S$ plays the role of the real line.

This was finally made precise by Morel and Voevodsky: In their trailblazing paper [MV99] from 1999 they developed a machinery, called the $\mathbb{A}^1$-homotopy theory of schemes. This machinery made it possible to apply the powerful tools of homotopy theory and algebraic topology in an algebraic setting, which soon led to spectacular applications such as the solution of the Block-Kato conjecture by Voevodsky.

Coming back to vector bundles, a naive hope one might have is that, in line with this idea, for a smooth scheme S the functor

$$\mathrm{Vect}\colon \mathrm{Sm}_{/S} \to \mathrm{Set},$$

which sends as smooth S-scheme X to the set of isomorphism classes of vector bundles over X, is $\mathbb{A}^1$-*invariant*. It turns out that this fails quite spectacularly: Even in the very simple case where $S = \mathrm{Spec}\,(k)$ is a field, the induced map $\mathrm{Vect}(\mathbb{P}^1_k) \to \mathrm{Vect}(\mathbb{P}^1_k \times \mathbb{A}^1)$ is not a bijection. However, there is a different invariant, derived from $\mathrm{Vect}(X)$, which does not see this issue.

The set of isomorphism classes of vector bundles over X namely admits the structure of a commutative monoid, given by the direct sum operation. Considering the group completion of this commutative monoid, which we denote by $K_0(X)$, it turns out that this construction is in fact $\mathbb{A}^1$-invariant when X is regular and noetherian. This leads us to the study of *algebraic K-theory*.

Algebraic K-theory had its origins in geometric topology and algebraic geometry. However, the real birth of algebraic K-theory took place in 1957, when Grothendieck defined the K-group of a subcategory of an abelian category $\mathcal{A}$. Taking $\mathcal{A}$ to be the category of all finite locally free sheaves over a regular noetherian scheme X, we get what we denoted by $K_0(X)$ above. It had its first prominent appearance in Grothendieck's reformulation of the

Riemann-Roch Theorem in [BS58] and since then continued to appear in many deep results and conjectures.

In the next years, the groups $K_1(R)$ and $K_2(R)$ of a ring R were defined by Bass and Milnor and the search for higher K-groups created a lot of research activity during the 1960s. The most important of the various constructions of higher K-groups that emerged during this period was Quillen's plus construction from 1969. Armed with his definition, Quillen was also able to perform one of the first important calculations of the subject: In [Qui72], he computed all higher K-groups of finite fields $\mathbb{F}_q$.

Soon after that, Quillen defined the higher algebraic K-theory of an *exact category* in his fundamental work [Qui73]. The $Q_\bullet$-construction, which he used to do so, gave rise to a K-theory space, whose fundamental groups are defined to be the K-groups. This work was later modified and extended by Waldhausen in [Wal85], where he introduced the $S_\bullet$-construction, which allowed to define higher K-groups of more general categories, today called *Waldhausen categories*.

Using Waldhausen's definition of higher algebraic K-theory, Thomason and Trobaugh in [TT07] were able to prove Zariski and Nisnevich descent results for the algebraic K-theory of a scheme X. This actually motivated some aspects of the construction of Morel and Voevodsky's $\mathbb{A}^1$-homotopy theory of schemes: Algebraic K-theory namely should fit into their framework and was therefore built in a way ensuring that in nice cases, i.e. when the base scheme is noetherian and regular, algebraic K-theory is an example of what they call a *motivic space*.

The goal of this thesis is to give a modern account of the above constructions and results, using the convenient language of ∞-categories. These tend to be an incredibly useful tool when trying to implement homotopical methods into other areas of mathematics. So, from a modern point of view, it is very natural to use them when describing the motivic homotopy theory of schemes and algebraic K-theory. The main content splits into two parts:

The first part, consisting of the first two chapters, is concerned with studying the motivic homotopy theory of a quasi-compact and quasi-separated scheme S. In the first chapter we will study the ∞-topos of Nisnevich sheaves of spaces over the site of smooth S-schemes. We will prove a few first important results, such as *Nisnevich excision*, which characterizes Nisnevich descent in terms of an excision property. We will also compare this to the construction of Morel and Voevodsky. To do so, we will show that, if our base scheme S is particularly nice (i.e. noetherian and of finite Krull dimension), the infinity topos we construct is *hypercomplete*, which implies that it agrees with the ∞-topos underlying the model category constructed in [MV99].

In the second chapter we will introduce the unstable motivic homotopy category $\mathcal{H}(S)$ of a quasi-compact and quasi-separated scheme S. We will deduce some basic results and then turn towards studying the functoriality of the construction $S \mapsto \mathcal{H}(S)$. The main goal of this chapter is to prove the so called *localization* or *gluing* theorem, which, roughly speaking, states that a motivic space $\mathcal{F} \in \mathcal{H}(S)$ may be glued together from its restriction to an open subscheme U and its complement $X \setminus U$.

In the second part of this thesis we will study the algebraic K-theory of schemes. In the third chapter, we will introduce the $S_\bullet$-construction and use it to define the algebraic K-theory of stable ∞-categories and then study the properties of this construction. The main goal will be to show that the algebraic K-theory functor takes certain *exact sequences* of stable ∞-categories to fiber sequences of K-theory spaces. We will then define the non-connective K-theory spectrum and deduce that it is a *localizing invariant*.

In the fourth chapter we will apply the methods of the third chapter in order to define and study the algebraic K-theory of schemes. The beginning will be concerned with setting the stage. We will construct appropriate derived ∞-categories of quasi-coherent $\mathcal{O}_X$-modules and *perfect complexes*. The algebraic K-theory of a scheme X will then be defined as the K-theory of the ∞-category of perfect complexes. We will then apply the results developed in the previous chapter to deduce that algebraic K-theory satisfies Nisnevich descent.

In the fifth chapter we will introduce the G-theory space of a noetherian scheme X and show that it is canonically equivalent to the K-theory space of X, if X is regular. Then Quillen's classical result about the $\mathbb{A}^1$-invariance of G-theory will imply that K-theory is in fact a motivic space, if X is regular and noetherian.

0.1. Notations and Conventions

0.2. Throughout this thesis we will constantly use the language and tools of higher category theory. More specifically, we will use quasi-categories as a model for $(\infty, 1)$-categories. In particular, we will use the term "∞-category" as a synonym for "quasi-category". Furthermore, we will mostly adopt the set-theoretical and terminological conventions used in [Lur09].

0.3. We will use the following notational conventions:

- We will write $\mathcal{S}$ for the ∞-category of *spaces*. To be more precise, this is the localization (in the ∞-categorical sense) of the 1-category of small simplicial sets, denoted by sSet, at the subcategory of weak equivalences.
- We will write Cat_∞ for the ∞-category of small ∞-categories.
- We will write $\mathcal{P}r^L$ for the ∞-category of presentable ∞-categories and small colimit preserving functors between them.
- For a small ∞-category C, we will write $\mathrm{Psh}(C) := \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S})$ for the ∞-category of presheaves on C.
- For a small ∞-category C, we will write

  $$h_C \colon C \to \mathrm{Psh}(C)$$

  for the ∞-categorical Yoneda embedding. Usually we will drop the index above and simply write h.

1. Nisnevich Sheaves

The goal of this chapter is to construct and study the ∞-topos of Nisnevich sheaves on the site of smooth schemes over a quasi-compact and quasi-separated scheme S. In section 1.1 we will introduce the Nisnevich topology on the category $\mathrm{Sm}_{/S}$ of smooth finitely presented

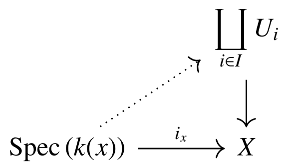
S-schemes. Section 1.2 will be devoted to formulating and proving Theorem 1.15, which characterizes Nisnevich sheaves in terms of an excision property. In section 1.3 we will show that, if our base scheme S is noetherian and of finite Krull dimension, then the ∞-topos of Nisnevich sheaves is hypercomplete (Theorem 1.29). This will imply that, in this case, the ∞-topos of Nisnevich sheaves that we consider agrees with the ∞-topos underlying the model category used in [MV99] to model the homotopy theory of Nisnevich $(\infty, 1)$-sheaves (see Remark 1.31). Then in section 1.4, we will discuss points of the Nisnevich topos and use the results from section 1.3 to deduce that the ∞-topos of Nisnevich sheaves has enough points (Theorem 1.46), if the base scheme S is noetherian and of finite Krull dimension.

1.1. The Nisnevich Topology

Definition 1.1. Let S be a quasi-compact and quasi-separated scheme. By $\mathrm{Sm}_{/S}$ we denote the category of schemes T over S such that the structure morphism $T \to S$ is smooth and of finite presentation. We define a pretopology on $\mathrm{Sm}_{/S}$ as follows:

A family of morphisms $\{U_i \to X\}_{i \in I}$ is a covering family of X if and only if

- the set I is finite,
- for every $i \in I$, the morphism $U_i \to X$ is étale and
- for every $x \in X$ and every solid diagram



  there exists a dotted arrow making the diagram commute. Here $k(x)$ denotes the residue field at x and i_x is the canonical morphism.

We call a morphism of schemes $Y \to X$ satisfying the above lifting property a *distinguished Nisnevich covering morphism*. It is easy to check that this indeed defines a pretopology on $\mathrm{Sm}_{/S}$.

We define the *Nisnevich topology* to be the topology on $\mathrm{Sm}_{/S}$ induced by the pretopology above (see Construction A.4).

Remark 1.2. Since a morphism of finite presentation is by definition quasi-compact and quasi-separated, it follows from [Sta20, Tag 01KV] and [Sta20, Tag 03GI] that any morphism $f \colon X \to Y$ in $\mathrm{Sm}_{/S}$ is quasi-compact and quasi-separated.

Examples 1.3.

i) Clearly every Zariski covering family is also a Nisnevich covering family.

ii) Let k be a field with char $k \neq 2$. Then the canonical morphism

$$p\colon \operatorname{Spec}\left(k[X,T,T^{-1}]/(X^2-T)\right) \to \operatorname{Spec}\left(k[T,T^{-1}]\right) = \mathbb{A}^1_k \setminus \{0\}$$

clearly is an étale covering. However, the canonical morphism

$$i_\eta\colon \operatorname{Spec}(k(T)) \to \mathbb{A}^1_k \setminus \{0\}$$

does not admit a lift along p. It follows that $\{p\}$ is an étale covering but not a Nisnevich covering family.

iii) We consider the morphism p from above and the inclusion of the open subscheme

$$j\colon \mathbb{A}^1_k \setminus \{0,1\} \to \mathbb{A}^1_k \setminus \{0\}.$$

Then the family $\{p, j\}$ is a Nisnevich covering family. This follows because the fiber of p over 1 is given by $\operatorname{Spec}(k) \amalg \operatorname{Spec}(k)$ and we therefore find a lift of the morphism $\operatorname{Spec}(k(1)) \to \mathbb{A}^1_k \setminus \{0\}$ along p.

In the case of S being a quasi-compact, quasi-separated scheme, one has the following alternative characterization:

Proposition 1.4. *Let X be a quasi-compact and quasi-separated scheme and let $\pi\colon Y \to X$ be an étale morphism. Then π is a distinguished Nisnevich covering morphism if and only if there is a sequence of finitely presented closed subschemes of X*

$$\emptyset = Z_n \subseteq \ldots \subseteq Z_1 \subseteq Z_0 = X$$

such that, for all $0 \leq i \leq n-1$, the induced map on pullbacks

$$(Z_i \setminus Z_{i+1}) \times_X Y \to Z_i \setminus Z_{i+1}$$

admits a section.

Proof: It is clear that a morphism admitting such a sequence is a distinguished Nisnevich covering morphism. For the converse implication, see [Hoy16]. □

Definition 1.5. In the situation of Proposition 1.4, we call a sequence

$$\emptyset = Z_n \subseteq \ldots \subseteq Z_1 \subseteq Z_0 = X$$

satisfying the above property a *splitting sequence* of π.

Remark 1.6. It is a well-known fact that, for any scheme S, the category of finite type schemes over S is essentially small. Thus, as a subcategory of an essentially small category, the category $\mathrm{Sm}_{/S}$ is essentially small, too.

We are now ready to introduce the main subject of study in this chapter:

Definition 1.7. Let S be scheme. We define $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$ to be the ∞-category of sheaves of spaces on $\mathrm{Sm}_{/S}$ with respect to the Nisnevich topology (see Construction A.5). In particular, there is a left exact localization functor

$$L_S^{\mathrm{Nis}}\colon\ \mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$$

which is left adjoint to the inclusion $i\colon\ \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S}) \hookrightarrow \mathrm{Psh}(\mathrm{Sm}_{/S})$. We will say that a morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ in $\mathrm{Psh}(\mathrm{Sm}_{/S})$ is a *Nisnevich-local equivalence* if $L_S^{\mathrm{Nis}}(\alpha)$ is an equivalence.

We will now discuss the functoriality of the above construction in the base scheme S.

1.8. Let $f\colon X \to S$ be a morphism of quasi-compact and quasi-separated schemes. Then we get an induced functor

$$- \times_S X\colon\ \mathrm{Sm}_{/S} \to \mathrm{Sm}_{/X}$$

given by pulling back along f. It is easy to check that this is in fact a morphism of sites. So by Proposition A.17, we get a geometric morphism of ∞-topoi

$$f^*\colon\ \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S}) \rightleftarrows \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/X}) : f_*,$$

where f_* is given by precomposing a sheaf $\mathcal{F}\colon\ \mathrm{Sm}_{/X} \to \mathcal{S}$ with the functor $- \times_S X$. If the morphism $f\colon X \to Y$ is smooth and of finite presentation, it follows from Example A.19 that f^* is given by precomposing with the forgetful functor $\mathrm{Sm}_{/X} \to \mathrm{Sm}_{/S}$ and has a further left adjoint

$$f_\sharp\colon\ \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/X}) \to \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S}).$$

In particular f^* preserves all small limits and colimits.

1.2. Nisnevich Excision

This section is devoted to proving Theorem 1.15, which provides a nice, simple way of determining whether a presheaf $\mathcal{F}$ in $\mathrm{Psh}(\mathrm{Sm}_{/S})$ is a Nisnevich sheaf.

Definition 1.9. A pullback diagram in $\mathrm{Sm}_{/S}$

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{\ j\ } & X \end{array}$$

is called an *elementary distinguished Nisnevich square* if

- the map j is an open immersion,
- the map p is étale and
- the induced map $V \times_X (X \setminus U) \to X \setminus U$ is an isomorphism, where we equip $X \setminus U$ with the induced reduced subscheme structure.

We will often simply call such a square a *Nisnevich square*.

We define the *Nisnevich cd-topology* on Sm_S to be the topology generated by the coverage (see Definition A.12) consisting of

– families of the form $\{j\colon U \to X, p\colon V \to X\}$ such that

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{\;j\;} & X \end{array}$$

is a Nisnevich square and

– the empty family as a covering family of the empty scheme.

Remark 1.10. In the above definition it does not matter which subscheme structure we put on $X \setminus U$. By this we mean that, given a Nisnevich square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{\;j\;} & X \end{array}$$

the induced morphism $V \times_X (X \setminus U) \to X \setminus U$ is an isomorphism for any closed subscheme structure on $X \setminus U$ if and only if it is an isomorphism for the reduced structure on $X \setminus U$. To see this, we consider the pullback square

$$\begin{array}{ccc} V \times (X \setminus U) & \longleftarrow & V \times (X/U)^{\mathrm{red}} \\ \downarrow & & \downarrow \\ X \setminus U & \longleftarrow & (X \setminus U)^{\mathrm{red}} \end{array}$$

(with any closed subscheme structure on $X \setminus U$) and note that the right vertical arrow is an isomorphism if the left one is. Conversely if the right vertical arrow is an isomorphism, the left vertical arrow is a universal homeomorphism, as the horizontal arrows are. Then the claim follows since an étale universal homeomorphism is an isomorphism (see [Bar10, Proposition 3.1]).

Example 1.11.

i) For $X \in \mathrm{Sm}_{/S}$ and $U, V \subseteq X$ two open subschemes of X such that $X = U \cup V$, the square

$$\begin{array}{ccc} U \cap V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \hookrightarrow & X \end{array}$$

is a Nisnevich square. We will refer to such a square as a *Zariski square*.

ii) Let k be a field with $\operatorname{char} k \neq 2$. Then the open immersion $j\colon \mathbb{A}^1_k \setminus \{1\} \hookrightarrow \mathbb{A}^1_k$ and the étale morphism

$$p\colon \mathbb{A}^1_k \setminus \{1,0\} \to \mathbb{A}^1_k$$
$$z \mapsto z^2$$

give rise to a Nisnevich square.

Proposition 1.12. *Let S be a quasi-compact and quasi-separated scheme. Then the Nisnevich cd-topology and the Nisnevich topology agree.*

Proof: The following argument is taken from [Voe08, Proposition 2.16]. It is easy to check that, for any Nisnevich square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

the induced map $U \amalg V \to X$ is a distinguished Nisnevich covering morphism, which shows that the Nisnevich topology is finer than the Nisnevich cd-topology. Furthermore, by definition both topologies contain the empty family as a covering of the empty set.

So let conversely $\{U_i \to X\}_{i \in I}$ be a Nisnevich covering family. Since the Zariski square

$$\begin{array}{ccc} \emptyset & \longrightarrow & X_{i_2} \\ \downarrow & & \downarrow \\ X_{i_1} & \longrightarrow & X_{i_1} \amalg X_{i_2} \end{array}$$

is a Nisnevich square, it follows inductively that the family

$$\left\{ U_i \hookrightarrow \coprod_j U_j \right\}_i$$

is a covering with respect to the Nisnevich cd-topology. So it suffices to see that the induced morphism $\pi\colon Y = \coprod_i U_i \to X$ is a covering morphism in the Nisnevich cd-topology. By Proposition 1.4, there is a splitting sequence

$$\emptyset = Z_n \subseteq \ldots \subseteq Z_1 \subseteq Z_0 = X$$

of π. By assumption, the pulled back map $Z_{n-1} \times_X Y \to Z_{n-1}$ is étale and has a section s, which therefore is an open immersion (see [Sta20, Tag 024T]). So $s(Z_{n-1}) \subseteq Z_{n-1} \times_X Y$ is open and hence its complement $T \subseteq Z_{n-1} \times_X Y$ is closed. Furthermore, the morphism $Z_{n-1} \times_X Y \to Y$ is a closed immersion, so the image of T in Y is closed. Its complement, say Y', is open in Y.

We claim that the induced square

$$\begin{array}{ccc} (X \setminus Z_{n-1}) \times_X Y' & \longrightarrow & Y' \\ \downarrow & & \downarrow p' \\ X \setminus Z_{n-1} & \xrightarrow{j'} & X \end{array}$$

is a distinguished Nisnevich square. For this we have to see that the induced morphism

$$\bar{p}\colon Y' \times_X Z_{n-1} \to Z_{n-1}$$

is an isomorphism. Note that the section $s\colon Z_{n-1} \to Y \times_X Z_{n-1}$ of the projection

$$Z_{n-1} \times_X Y \to Z_{n-1}$$

restricts to a section $s'\colon Z_{n-1} \to Y' \times_X Z_{n-1}$ of the projection

$$Y' \times_X Z_{n-1} \to Z_{n-1}.$$

Furthermore, by construction we have

$$\begin{aligned} Y' \times_X Z_{n-1} &\cong (Y \setminus T) \times_X Z_{n-1} \\ &\cong (Y \times_X Z_{n-1}) \setminus (T \times_X Z_{n-1}) \\ &\cong (Y \times_X Z_{n-1}) \setminus T \\ &= s(Z_{n-1}). \end{aligned}$$

It follows that s' is a surjective open immersion, thus an isomorphism, so the above square indeed is a distinguished Nisnevich square.

Moreover, the pullback of p along p' has a section and the pullback along j' has a splitting sequence of length $n-1$. So the claim follows by induction and the fact that a map with a splitting sequence of length 1 splits and is thus covering. □

Definition 1.13. Let $\mathcal{F} \in \mathrm{Psh}(\mathrm{Sm}_{/S})$ be a presheaf. We will say that $\mathcal{F}$ satisfies *Nisnevich excision* if $\mathcal{F}(\emptyset) \simeq *$ and, for any Nisnevich square

$$\begin{array}{ccc} U \times_X V & \xrightarrow{j'} & V \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{j} & X \end{array}$$

the induced square

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{p^*} & \mathcal{F}(V) \\ \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j'^*} \\ \mathcal{F}(U) & \xrightarrow{p'^*} & \mathcal{F}(U \times_X V) \end{array}$$

is a pullback square in $\mathcal{S}$.

Notation 1.14. If $U_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to C$ is a simplicial object in any ∞-category, we will write

$$|U_\bullet|$$

for its colimit in C (if it exists). We will call $|U_\bullet|$ the *geometric realization* of $U_\bullet$.

Theorem 1.15 (Voevodsky)**.** *Let S be a quasi-compact and quasi-separated scheme and let $\mathcal{F} \in \mathrm{Psh}(\mathrm{Sm}_{/S})$ be a presheaf. Then $\mathcal{F}$ is a Nisnevich sheaf if and only if it satisfies Nisnevich excision.*

Proof: We start with the following general observation: Let

$$\begin{array}{ccc} U \times_X V & \xrightarrow{j'} & V \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{j} & X \end{array} \tag{1}$$

be a Nisnevich square and let

$$\mathcal{U} = \{U \xrightarrow{j} X, V \xrightarrow{p} X\}$$

be the corresponding Nisnevich covering family. By R we denote the sieve generated by $\mathcal{U}$. For simplicity, we write

$$V_\bullet = \check{C}_\bullet(p) \qquad \text{and} \qquad U_\bullet = \check{C}_\bullet(p')$$

for the Čech-nerves of p and p' (see Construction A.9). We now claim that the canonical map

$$\phi\colon |V_\bullet| \amalg_{|U_\bullet|} h(U) \to R$$

is an equivalence in $\mathrm{Psh}(\mathrm{Sm}_{/S})$. It follows from Lemma A.10 that all presheaves in discussion are 0-truncated. This colimit is computed objectwise and the morphism

$$|U_\bullet|(T) \to h(T)$$

is an injective map of sets for every $T \in \mathrm{Sm}_{/S}$. Hence the homotopy pushout

$$|V_\bullet|(T) \amalg_{|U_\bullet|(T)} h(U)(T)$$

in $\mathcal{S}$ is just the ordinary pushout in the category of sets. So we simply have to show that the map of sets $\phi(X)$ is bijective for every $X \in \mathrm{Sm}_{/S}$.

Surjectivity is obvious. For injectivity, we note that, since both canonical morphisms $|V_\bullet|(T) \to R(T)$ and $h(U)(T) \to R(T)$ are injective, it suffices to see that any $f \in |V_\bullet|(T)$ and $g \in h(U)(T)$ with $\phi(T)(f) = \phi(T)(g)$ agree in the pushout. So let us pick such an $f\colon T \to X$ that factors though p and a $g\colon T \to U$ such that $j \circ g = f$. Therefore, we get an induced morphism $h\colon T \to U \times_X V$ such that $p' \circ h = g$ and $p \circ j' \circ h = f$. In particular, the morphism g gives rise to an element in $|U_\bullet|(T)$ that maps to g and f, respectively, under the canonical morphisms. It follows that f and g represent the same object in the pushout which proves the claim.

So, for any presheaf $\mathcal{F}$, the canonical morphism

$$\begin{aligned} \lim_{W \to X \in R} \mathcal{F}(W) &\simeq \\ \mathrm{map}_{\mathrm{Psh}(\mathrm{Sm}_{/S})}(R, \mathcal{F}) &\xrightarrow{\phi^*} \mathrm{map}_{\mathrm{Psh}(\mathrm{Sm}_{/S})}\left(|V_\bullet| \amalg_{|U_\bullet|} h(U), \mathcal{F}\right) \\ &\simeq \lim \mathcal{F}(V_\bullet) \times_{\lim \mathcal{F}(U_\bullet)} \mathcal{F}(U) \end{aligned}$$

is an equivalence.

Now let us assume that $\mathcal{F}$ is a Nisnevich sheaf on $\mathrm{Sm}_{/S}$. It is clear that $\mathcal{F}(\emptyset) \simeq *$, since the empty sieve covers $\emptyset$. Also, for a square like (1), the family

$$\mathcal{U} = \{U \xrightarrow{j} X, V \xrightarrow{p} X\}$$

is a Nisnevich covering family. It follows that, using the notations from above, we have a canonical equivalence

$$\mathcal{F}(X) \simeq \lim_{W \to X \in R} \mathcal{F}(W) \simeq \lim \mathcal{F}(V_\bullet) \times_{\lim \mathcal{F}(U_\bullet)} \mathcal{F}(U).$$

So let us consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{F}(X) & \longrightarrow & \lim \mathcal{F}(V_\bullet) & \longrightarrow & \mathcal{F}(V) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \lim \mathcal{F}(U_\bullet) & \longrightarrow & \mathcal{F}(U \times_X V) \end{array}$$

where the right horizontal maps are given by the projections. We wish to show that the outer square is cartesian. Because we know that the left square is cartesian, it suffices to show that the right square is, too. The latter is the limit of the squares

$$\begin{array}{ccc} \mathcal{F}(V_n) & \longrightarrow & \mathcal{F}(V) \\ \downarrow & & \downarrow \\ \mathcal{F}(U_n) & \longrightarrow & \mathcal{F}(U \times_X V) \end{array}$$

where the horizontal arrows are induced by the diagonals. So it suffices to see that these are pullbacks for all n. To this end, we consider the commutative square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow \Delta \\ U_n = U \times_X V \times_X \ldots \times_X V & \xrightarrow{\gamma} & V \times_X \ldots \times_X V = V_n \end{array} \tag{2}$$

We observe that Δ is an open immersion as p is étale. Moreover, the morphism γ is an open immersion as well since it sits in the pullback square

$$\begin{array}{ccc} U_n & \longrightarrow & V_n \\ \downarrow & & \downarrow \\ U & \xrightarrow{j} & X \end{array}$$

and j is an open immersion. Furthermore, the diagram

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow \Delta \\ U_n & \xrightarrow{\gamma} & V_n \\ \downarrow & & \downarrow \pi \\ U & \longrightarrow & X \end{array}$$

shows that (2) is a pullback square as well. Now, for a point $i_x\colon \operatorname{Spec}(k(x)) \to V_n$, such that i_x does not factor through U_n, it follows that $\pi \circ i_x$ does not factor through U. So $\pi \circ i_x$ factors through $X \setminus U$ (with the reduced subscheme structure). But, because (1) is a distinguished Nisnevich square, the morphism Δ induces an isomorphism

$$X \setminus U \cong (X \setminus U) \times_X V \xrightarrow{(X\setminus U)\times_X \Delta} (X \setminus U) \times_X V_n.$$

So it follows that i_x factors through V. If we identify U_n and V with the corresponding open subschemes of V_n, we have just seen that $V_n = U_n \cup V$ and $U \times_X V = U_n \cap V$. Let S denote the sieve generated by the Nisnevich covering

$$\{U_n \xrightarrow{\gamma} V_n, V \xrightarrow{\Delta} V_n\}.$$

Using similar methods as above, one easily sees that the canonical morphism

$$h(U_n) \amalg_{h(U\times_X V)} h(V) \to S$$

is an equivalence in $\mathrm{Psh}(\mathrm{Sm}_{/S})$. Since $\mathcal{F}$ is a sheaf, we get that the canonical morphism

$$\mathcal{F}(V_n) \xrightarrow{\simeq} \lim_{W \to V_n \text{ in } S} \mathcal{F}(W) \xrightarrow{\simeq} \mathcal{F}(U_n) \times_{\mathcal{F}(U\times_X V)} \times \mathcal{F}(V)$$

is an equivalence, which proves the claim.

Now let $\mathcal{F}$ conversely be a presheaf satisfying Nisnevich excision. Using Proposition 1.12 and Proposition A.13, it suffices to see that, for a square like (1) and the sieve R generated by

$$\{U \xrightarrow{j} X, V \xrightarrow{p} X\},$$

the canonical morphism

$$\mathcal{F}(X) \to \lim_{W \to X \text{ in } R} \mathcal{F}(W)$$

is an equivalence and that $\mathcal{F}$ satisfies the sheaf condition for the empty covering of $\emptyset$. The latter holds as, by assumption, we have $\mathcal{F}(\emptyset) \simeq *$. For the former, we consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{F}(X) & \longrightarrow & \lim \mathcal{F}(V_\bullet) & \longrightarrow & \mathcal{F}(V) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \lim \mathcal{F}(U_\bullet) & \longrightarrow & \mathcal{F}(U \times_X V) \end{array}$$

and, using the observation from the beginning of the proof, we have to show that the left square is a pullback square. Again, the right square is the limit of the squares

$$\begin{array}{ccc} \mathcal{F}(V_n) & \longrightarrow & \mathcal{F}(V) \\ \downarrow & & \downarrow \\ \mathcal{F}(U_n) & \longrightarrow & \mathcal{F}(U) \end{array}$$

and we have seen above that the squares

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow \Delta \\ U_n & \xrightarrow{\gamma} & V_n \end{array}$$

are distinguished Nisnevich (even Zariski) squares for all n. So, since $\mathcal{F}$ satisfies Nisnevich excision, it follows that the right square above is cartesian and so is the outer rectangle since (1) was assumed to be a Nisnevich square. Hence the claim follows. ◻

Remark 1.16. From [Sta20, Tag 03O3] and [Sta20, Tag 03PH] it follows that the Nisnevich topology on $\mathrm{Sm}_{/S}$ is subcanonical. Hence the above theorem shows that, for a Nisnevich square as above, the induced square

$$\begin{array}{ccc} h(U \times_X V) & \longrightarrow & h(U) \\ \downarrow & & \downarrow \\ h(V) & \longrightarrow & h(X) \end{array}$$

is a pushout square in the ∞-category $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$.

1.3. Hypercompleteness

Our next goal is to prove that the infinity topos $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$ is hypercomplete if S is noetherian and has finite Krull dimension. For this, we will follow [Lur11, §2].

Definition 1.17. We denote by S^{Nis} the full subcategory of $\mathrm{Sm}_{/S}$ spanned by those schemes that are étale over S. We equip this category with the topology induced by the Nisnevich topology. We denote the corresponding ∞-topos of sheaves of spaces by S_{Nis}.

Remark 1.18. Just like for $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$, we have that the topology on S^{Nis} is generated by the analogous version of the coverage in Definition 1.9. We also get an corresponding version of Theorem 1.15 for presheaves on S^{Nis}.

Definition 1.19.

i) Let $\mathcal{F} \in \mathrm{Psh}(S^{\mathrm{Nis}})$ be a presheaf equipped with a map $\gamma\colon \mathcal{F} \to h(X)$ for $X \in S^{\mathrm{Nis}}$. Then, for any morphism $f\colon U \to X$ in S^{Nis}, we define $\mathcal{F}_f(U)$ to be the pullback in

$$\begin{array}{ccc} \mathcal{F}_f(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ * & \xrightarrow{f} & \mathrm{map}_{S^{\mathrm{Nis}}}(U, X) \end{array}$$

ii) Let X be $\in S^{\mathrm{Nis}}$ and let $x \in X$ be a point. Then a *Nisnevich neighbourhood* of x is an étale map $g\colon U \to X$ together with a point $u \in U$ such that $g(u) = x$ and such that the induced map on residue fields $k(x) \to k(u)$ is an isomorphism.

iii) We will say that $\gamma\colon \mathcal{F} \to h(X)$ is *weakly* 0*-connective*, if for every $f\colon U \to X$ in S^{Nis} and all $x \in U$ of height 0, there is a Nisnevich neighbourhood $g\colon U' \to U$ of x such that $\mathcal{F}_{fg}(U')$ is non-empty.

For $n > 0$, we will say that γ is *weakly n-connective* if:

a) For every $f\colon U \to X$ in S^{Nis} and all $x \in U$ of height at most n, there is a Nisnevich neighbourhood $g\colon U' \to U$ of x such that $\mathcal{F}_{fg}(U')$ is non-empty.

b) For any pair of maps $h(U) \to \mathcal{F}$ and $h(V) \to \mathcal{F}$ the induced morphism

$$h(U) \times_{\mathcal{F}} h(V) \to h(U) \times_{h(X)} h(V) = h(U \times_X V)$$

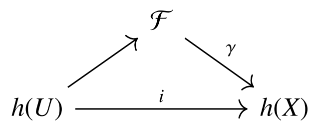
is weakly $(n-1)$-connective.

By convention, we will say that any morphism is weakly (-1)-connective.

We will defer the proof of the following lemma to a later section, after we have introduced the notion of stalks of a Nisnevich sheaf (see the end of section 1.4).

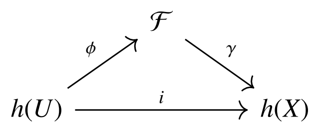
Lemma 1.20. *Let $\gamma\colon \mathcal{F} \to h(X)$ be an n-connective morphism in the ∞-topos* $\text{Sh}_{\text{Nis}}(\text{Sm}_{/S})$. *Then γ is weakly n-connective as a morphism in* $\text{Psh}(S^{\text{Nis}})$.

Lemma 1.21. *Let S be a noetherian scheme, let $\gamma\colon \mathcal{F} \to h(X)$ be weakly n-connective and assume that $\mathcal{F}$ is a Nisnevich sheaf. Then there is a finite set of points $x_1, ..., x_m \in X$ of height $> n$ and a commutative diagram*



where i is induced by the inclusion of the open subscheme $U = X \setminus \bigcup_i \overline{\{x_i\}}$.

Proof: We will proceed by induction on n. If $n = -1$, we can choose $x_1, ..., x_m$ to be the generic points of X. These are finitely many, since X is noetherian, as S is. Then $U = \emptyset$ and we clearly get a diagram as desired. Now let $n \geq 0$. Since γ is in particular weakly $(n-1)$-connective, the induction hypothesis implies that there are points $x_1, ..., x_m$ of height $\geq n$ and a commutative triangle



where $U = X \setminus \bigcup_i \overline{\{x_i\}}$. After reordering the x_i's, we may assume that there is a $0 \leq k \leq m$ such that $x_1, ..., x_k$ have height n and $x_{k+1}, ..., x_m$ have height $> n$. We may also assume that the x_i's are chosen in a way such that k is minimal. Our goal is to show that $k = 0$. Let us assume that $k \neq 0$, so x_1 has height n. Since γ is weakly n-connective, there is a Nisnevich neighbourhood $f\colon (X', x') \to (X, x_1)$ of x_1 such that $\mathcal{F}_f(X')$ is non-empty. Thus there is a commutative triangle

$$\begin{array}{ccccc} & & \mathcal{F} & & \\ & \overset{\psi}{\nearrow} & & \overset{\gamma}{\searrow} & \\ h(X') & & \xrightarrow{\quad f \quad} & & h(X) \end{array}$$

and we may shrink X' if necessary to assume that

$$f(X') \subseteq X \setminus \bigcup_{i=2}^{m} \overline{\{x_i\}}.$$

We consider the pullback square

$$\begin{array}{ccc} f^{-1}(\overline{\{x_1\}}) & \hookrightarrow & X' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \overline{\{x_1\}} & \hookrightarrow & X \end{array}$$

where we endow the closed subscheme $\overline{\{x_1\}}$ with the reduced structure. Also f' is étale and induces an isomorphism $k(x_1) \to k(x')$ on residue fields and thus splits at the generic point of $\overline{\{x_1\}}$. Since $f'\colon f^{-1}(\overline{\{x_1\}}) \to \overline{\{x_1\}}$ is finitely presented, it follows that there is an open subscheme $Q \subseteq \overline{\{x_1\}}$ and a morphism $s\colon Q \to f^{-1}(\overline{\{x_1\}})$ such that the diagram

$$\begin{array}{ccc} & & f^{-1}(\{\overline{x_1}\}) \\ & \overset{s}{\nearrow} & \downarrow{\scriptstyle f'} \\ Q & \hookrightarrow & \overline{\{x_1\}} \end{array}$$

commutes. Since f' is étale, the image $s(Q) \subseteq f^{-1}(\{\overline{x_1}\})$ is open. We can thus find a $V \subseteq X'$ open such that

$$V \cap f^{-1}(\{\overline{x_1}\}) = s(Q).$$

Now our given maps $\phi\colon h(U) \to \mathcal{F}$ and $\psi|_V : h(V) \to \mathcal{F}$ induce a morphism

$$h(U) \times_{\mathcal{F}} h(V) \to h(U \times_X V),$$

that is weakly $(n-1)$-connective, as γ is n-connective. For simplicity, we will write $U' = U \times_X V$. By applying the induction hypothesis, we get points $y_1, ..., y_l \in U'$ of height greater or equal than n and a commutative triangle

$$\begin{array}{ccccc} & & h(U) \times_{\mathcal{F}} h(V) & & \\ & \overset{\alpha}{\nearrow} & & \searrow & \\ h(W) & & \longrightarrow & & h(U') \end{array}$$

where $W = U' \setminus \bigcup_i \overline{\{y_i\}}$. Furthermore, we observe that in the above we may replace X' by

$$X' \setminus \bigcup_i \overline{\{y_i\}},$$

because $y_i \neq x'$ and all the y_i have height n, so $x' \notin \overline{\{y_i\}}$, since x' has height n by [Sta20, Tag 0AFF]. It follows that we may assume $W = U'$. Then α induces a commutative square

$$\begin{array}{ccc} h(U') & \longrightarrow & h(U) \\ \big\downarrow & & \big\downarrow \\ h(V) & \longrightarrow & \mathcal{F} \end{array} \tag{1}$$

by the universal property of the pullback. We claim that the pullback square

$$\begin{array}{ccc} U' & \longrightarrow & V \\ \big\downarrow & & \big\downarrow {\scriptstyle f|_V} \\ U & \longrightarrow & U \cup f(V) \end{array}$$

is a distinguished Nisnevich square. This follows, as

$$\begin{aligned} (U \cup f(V)) \setminus U &= (X \setminus U) \cap f(V) \\ &= \bigcup_{i=1}^{m} \overline{\{x_i\}} \cap f(V) \\ &= f(V) \cap \overline{\{x_1\}}, \end{aligned}$$

where the last equality holds, since, by construction,

$$f(V) \subseteq f(X') \subseteq X \setminus \bigcup_{i=2}^{m} \overline{\{x_i\}}.$$

But we have constructed V in such a way that the induced morphism

$$V \cap f^{-1}(\overline{\{x_1\}}) = s(Q) \to Q = f(V) \cap \overline{\{x_1\}}$$

is an isomorphism. By Remark 1.16, the square (1) thus gives rise to a morphism

$$\Phi\colon h(U \cup f(V)) \to \mathcal{F}$$

which also makes the diagram

$$\begin{array}{ccccc} & & \mathcal{F} & & \\ & {\scriptstyle \Phi}\nearrow & & \searrow{\scriptstyle \gamma} & \\ h(U \cup f(V)) & & \xrightarrow{\text{inclusion}} & & h(X) \end{array}$$

commute. Now let us denote by $t_1, ..., t_a$ all generic points of $\overline{\{x_1\}} \setminus f(V)$. Observe that, since $x_1 \in f(V)$, we have $t_i \neq x_1$ for all i. Then it follows that

$$U \cup f(V) = X \setminus \left(\bigcup_{i=2}^{m} \overline{\{x_i\}} \cup \bigcup_{j=1}^{a} \overline{\{t_i\}} \right)$$

because

$$\begin{aligned} X \setminus (U \cup f(V)) &= \left(\bigcup_{i=1}^{m} \overline{\{x_i\}} \right) \cap (X \setminus f(V)) \\ &= \left(\bigcup_{i=2}^{m} \overline{\{x_i\}} \right) \cup \left(\overline{\{x_1\}} \setminus f(V) \right) \\ &= \left(\bigcup_{i=2}^{m} \overline{\{x_i\}} \cup \bigcup_{j=1}^{a} \overline{\{t_i\}} \right). \end{aligned}$$

Finally we see that, since $t_i \in \overline{\{x_1\}}$ and $t_i \neq x_1$, we have that $\mathrm{ht}(t_i) > \mathrm{ht}(x_1) = n$, which contradicts the minimality of k. □

Let us quickly recall the following definitions ([Lur09, Def. 7.2.1.1 and 7.2.1.8]):

Definition 1.22.

i) An ∞-topos $\mathcal{X}$ is said to have *homotopy dimension* $\leq n$, if every n-connective object admits a global section.

ii) An ∞-topos $\mathcal{X}$ is said to be *locally of homotopy dimension* $\leq n$, if there is a collection of objects $\{U_\alpha\}_\alpha$ that generate $\mathcal{X}$ under colimits and such that $\mathcal{X}/U_\alpha$ is of homotopy dimension $\leq n$ for every α.

The main result is then the following ([Lur09, Proposition 7.2.1.10, Corollary 7.2.1.12]):

Theorem 1.23. *If $\mathcal{X}$ is locally of finite homotopy dimension, then Postnikov towers converge in $\mathcal{X}$. In particular, the ∞-topos $\mathcal{X}$ is hypercomplete.*

We will now apply this to our given situation:

Proposition 1.24. *Let S be a noetherian scheme of finite Krull dimension. Then the homotopy dimension of S_{Nis} is smaller or equal than the Krull dimension of S.*

Proof: Let n be the Krull dimension of X and let $\mathcal{F} \in S_{\mathrm{Nis}}$ be an n-connective object. This means that the canonical morphism $\gamma\colon \mathcal{F} \to h(S)$ is n-connective and, by Lemma 1.20, is weakly n- connective. So, by Lemma 1.21, we get that γ has a section, since S does not contain any points of height greater than n. □

Theorem 1.25. *If S is noetherian and has finite Krull dimension n, then S_{Nis} is locally of homotopy dimension $\leq n$. In particular, Postnikov towers converge in S_{Nis}. Thus S_{Nis} is a hypercomplete ∞-topos.*

Proof: Note that S_{Nis} is generated under colimits by the representable objects $h(U)$ for $U \in S^{\mathrm{Nis}}$. We claim that, for every $U \in S^{\mathrm{Nis}}$, one has a canonical equivalence of ∞-topoi

$$S_{\mathrm{Nis}}/h(U) \simeq U_{\mathrm{Nis}}.$$

This follows from [GV72, p.295] for the underlying 1-topoi of 0-truncated objects and thus the claim follows since both topoi in discussion are 1-localic (see [Lur09, §6.4.5]). Furthermore, the scheme U is also noetherian of Krull dimension at most n by [Sta20, Tag 0AFF], as U is étale over S. So, by Proposition 1.24, it follows that $S_{\mathrm{Nis}}/h(U)$ has homotopy dimension $\leq n$, as desired. □

We now aim at extending the above result from S_{Nis} to $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$.

1.26. For a scheme U, the inclusion

$$i_U : U^{\mathrm{Nis}} \to \mathrm{Sm}_{/U}$$

is a morphism of sites and thus the functor

$$\mathfrak{c}^{i_U} : \mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathrm{Psh}(U^{\mathrm{Nis}})$$

given by precomposition with i_U restricts to a geometric morphism

$$j_{U*} : \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/U}) \to U_{\mathrm{Nis}}.$$

Proposition 1.27. *The functor* $\mathfrak{c}^{i_U} : \mathrm{Psh}(\mathrm{Sm}_{/U}) \to \mathrm{Psh}(U^{\mathrm{Nis}})$ *preserves Nisnevich-local equivalences.*

Proof: It suffices to see that, for any $X \in \mathrm{Sm}_{/U}$ and any Nisnevich covering sieve $R \to h(X)$, the induced morphism $\mathfrak{c}^{i_U}(R \to h(X))$ is a Nisnevich-local equivalence. Since colimits in ∞-topoi are universal, it is enough to show that, for any $Y \in U^{\mathrm{Nis}}$ and any morphism $\psi : h(Y) \to \mathfrak{c}^{i_U}(h(X))$, the pulled back morphism

$$W := \mathfrak{c}^{i_U}(R) \times_{\mathfrak{c}^{i_U}(h(X))} h(y) \to h(y) \tag{1}$$

is a Nisnevich covering sieve. Since R is a Nisnevich covering sieve, there is a collection of morphisms $f_i : U_i \to X$ in R such that the induced morphism

$$\coprod_{i \in I} U_i \to X$$

is a distinguished Nisnevich covering morphism. By the Yoneda lemma, the morphism ψ is given by a $\phi : Y \to X$ over U. Unwinding the definitions, we see that, for $T \in U^{\mathrm{Nis}}$, the set $W(T)$ is given by all morphisms $g : T \to Y$ in U^{Nis} such that the composition

$$T \xrightarrow{g} Y \xrightarrow{\phi} X$$

lies in R. It follows that W contains all $f'_i : U_i \times_X Y \to Y$, which are given by pulling back the f_i along ϕ. Finally, since

$$\coprod_{i \in I} U_i \times_X Y \to Y$$

is an elementary distinguished covering morphism, it follows that W is a Nisnevich covering sieve and in particular (1) is a Nisnevich-local equivalence. □

Corollary 1.28. *The functor* $j_{U*}\colon \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/U}) \to U_{\mathrm{Nis}}$ *preserves all colimits.*

Proof: Note that j_{U*} is given by the composite

$$\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/U}) \xrightarrow{j} \mathrm{Psh}(\mathrm{Sm}_{/U}) \xrightarrow{\mathfrak{c}^{i_U}} \mathrm{Psh}(U^{\mathrm{Nis}}) \xrightarrow{L} U_{\mathrm{Nis}}.$$

Here, the morphism j denotes the inclusion of the full subcategory $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/U}) \subseteq \mathrm{Psh}(\mathrm{Sm}_{/U})$ and L denotes the left adjoint of the inclusion of U_{Nis} into $\mathrm{Psh}(U^{\mathrm{Nis}})$. Now let $D\colon I \to \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/U})$ be a small diagram. We will write $C \in \mathrm{Psh}(\mathrm{Sm}_{/U})$ for the colimit of $j \circ D$. Then the canonical morphism

$$\alpha\colon C \to j(\mathrm{colim}_i D(i))$$

in $\mathrm{Psh}(\mathrm{Sm}_{/S})$ is clearly a Nisnevich-local equivalence. By Proposition 1.27, the induced morphism $\mathfrak{c}^{i_U}(\alpha)$ is a Nisnevich-local equivalence as well. It follows that the canonical morphism

$$\begin{aligned} \mathrm{colim}_i\, j_{U*}(D(i)) &\simeq \\ \mathrm{colim}_i L\mathfrak{c}^{i_U}(j(D(i))) &\simeq \\ L\mathfrak{c}^{i_U}(C) &\xrightarrow{L\mathfrak{c}^{i_U}(\alpha)} L\mathfrak{c}^{i_U}(j(\mathrm{colim}_i D(i)) \\ &\simeq\ j_{U*}(\mathrm{colim}_i D(i)) \end{aligned}$$

is an equivalence, which proves the claim. □

Corollary 1.29. *Let S be noetherian and of finite Krull dimension. Then Postnikov towers converge in the ∞-topos $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$. In particular, the ∞-topos $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$ is hypercomplete.*

Proof: Recall that, for any $f\colon U \to S$ in $\mathrm{Sm}_{/S}$, we get an induced functor $f^*\colon \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S}) \to \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/U})$. Now we observe that a morphism α in $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$ is an equivalence if and only if $i_{U*} \circ f^*$ is an equivalence for every $f\colon U \to S$ in $\mathrm{Sm}_{/S}$. Since both j_{U*} and f^* preserve all limits and colimits by Corollary 1.28 and thus commute with truncations by [Lur09, Proposition 5.5.6.28], the claim follows from [Lur09, Proposition 5.5.6.26]. □

Remark 1.30. The fact that $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$ is hypercomplete is equivalent to saying that any sheaf $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$ automatically satisfies descent with respect to all hypercoverings in $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$ (see Theorem A.23).

Remark 1.31. Classically, one presents the ∞-topos $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$ by the Joyal-Jardine Nisnevich-local model structure on the category of simplicial presheaves $\mathrm{Fun}(\mathrm{Sm}_{/S}, \mathrm{sSet})$ (see [AE17, Warning 3.49]). It follows from Corollary 1.29 and [Lur09, Proposition 6.5.2.14] that the ∞-topos underlying this simplical model category is equivalent to the ∞-topos $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$ constructed above, if S is noetherian and of finite Krull dimension.

1.4. Stalks in the Nisnevich Topology

Definition 1.32.

i) We will define a category $\mathrm{Sm}_{/S_*}$ as follows: The objects are pairs (U,u), where U is an object of $\mathrm{Sm}_{/S}$ and u is a point of the underlying topological space of U. A morphism $f\colon (U,u) \to (V,v)$ is a morphism $f\colon U \to V$ in $\mathrm{Sm}_{/S}$ such that $f(u) = v$. We call an object in $\mathrm{Sm}_{/S_*}$ a *pointed smooth S-scheme*.

ii) Let (U,u) be a pointed smooth S-scheme. We define the category $\mathrm{Nbhd}(U,u)$ to be the full subcategory of the slice

$$\mathrm{Sm}_{/S_*}/(U,u)$$

on the Nisnevich neighbourhoods of (U,u).

Proposition 1.33. *Using the notations from above, the category* $\mathrm{Nbh}(U,u)$ *is cofiltered.*

Proof: This is clear, since the inclusion $\mathrm{Nbhd}(U,u) \hookrightarrow \mathrm{Sm}_{/S_*}/(U,u)$ preserves pullbacks and therefore, the category $\mathrm{Nbhd}(U,u)$ has all pullbacks. □

Definition 1.34.

i) Let $(U,u) \in \mathrm{Sm}_{/S_*}$ be a pointed smooth S-scheme. Note that we have a canonical functor $j\colon \mathrm{Nbhd}(U,u)^{\mathrm{op}} \to \mathrm{Sm}_{/S}^{\mathrm{op}}$. We define the *stalk functor* at (U,u) to be the composite

$$(-)_u\colon \mathrm{Psh}(\mathrm{Sm}_{/S}) \xrightarrow{-\circ j} \mathrm{Fun}(\mathrm{Nbhd}(U,u)^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\mathrm{colim}} \mathcal{S}.$$

More informally, the functor $(-)_u$ is given by the assignment

$$\mathcal{F} \mapsto \operatorname*{colim}_{(V,v)\in \mathrm{Nbhd}(U,u)^{\mathrm{op}}} \mathcal{F}(V).$$

ii) We call a morphism f in $\mathrm{Psh}(\mathrm{Sm}_{/S})$ a *stalkwise equivalence* if f_u is an equivalence for all $(U,u) \in \mathrm{Sm}_{/S_*}$.

Lemma 1.35. *Every Nisnevich-local equivalence in* $\mathrm{Psh}(\mathrm{Sm}_{/S})$ *is a stalkwise equivalence.*

Proof: It suffices to see that any Nisnevich covering sieve $\varphi\colon R \to h(x)$ is a stalkwise equivalence. So let $(U,u) \in \mathrm{Sm}_{/S_*}$ and let us write k for the residue field of U at u. Since 0-truncated spaces are stable under filtered colimits in the ∞-category of spaces, we conclude that the stalk $h(x)_u$ is given by the colimit of sets

$$h(x)_u \cong \operatorname*{colim}_{(V,v)\in \mathrm{Nbhd}(U,u)^{\mathrm{op}}} \mathrm{map}_{\mathrm{Sm}_{/S}}(V,X).$$

As R is also a 0-truncated object of $\mathrm{Psh}(\mathrm{Sm}_{/S})$, we get an analogous formula for the stalk R_u. Furthermore, filtered colimits of injective maps of sets are injective, so it follows that the induced map

$$\varphi_u\colon R_u \to h(x)_u$$

is injective and it remains to show surjectivity. Note that an element $a \in h(x)_u$ is represented by a commutative triangle

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & X \\ & \searrow \quad \swarrow & \\ & S & \end{array} \tag{1}$$

for some $(V, v) \in \mathrm{Nbhd}(U, u)$. Since R is a Nisnevich covering sieve, there is a finite collection of morphisms $\{f_i \colon U_i \to X\}_{i \in I}$ in R such that the induced morphism

$$\pi \colon \coprod_i U_i \to X$$

is a distinguished Nisnevich covering morphism. Now $\alpha(v)$ is a k-point of X and since π is a Nisnevich-covering morphism, it follows that there is an $i_0 \in I$ and a map $\mathrm{Spec}\,(k) \to U_{i_0}$ lifting $\alpha(v)$. By pulling back along α, we get a map

$$U_{i_0} \times_X V \to V$$

and a point $x \colon \mathrm{Spec}\,(k) \to U_{i_0} \times_X V$ lying over v. Thus $(U_{i_0} \times_X V, x)$ is a Nisnevich neighbourhood of (U, u) equipped with a map to (V, v). We write α' for the composite

$$U_{i_0} \times_X V \xrightarrow{\mathrm{pr}_2} V \xrightarrow{\alpha} X$$

and then the induced diagram

$$\begin{array}{ccc} U_{i_0} \times_X V & \xrightarrow{\alpha'} & X \\ & \searrow \quad \swarrow & \\ & S & \end{array} \tag{2}$$

and (1) above represent the same element in $h(x)_u$. Finally, since α' factors through U_{i_0}, the element represented by (2) lies in the image of φ_u and we get the claim. □

Using essentially the same argument as in the proof of Corollary 1.28, we get the following:

Corollary 1.36. *The composite*

$$\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S}) \to \mathrm{Psh}(\mathrm{Sm}_{/S}) \xrightarrow{(-)_u} \mathcal{S}$$

preserves all small colimits.

Remark 1.37.

i) We will abuse notation and denote the above composite by $(-)_u$ as well.

ii) Since filtered colimits commute with finite limits in $\mathcal{S}$, it follows from Corollary 1.36 that the stalk functor $(-)_u$ preserves all colimits and finite limits. Thus the stalk functor defines a geometric morphism

$$u_* \colon \mathcal{S} \to \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S}),$$

so in other words a point of the ∞-topos $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$.

We will now relate stalk functors to the local rings in the Nisnevich topology:

Definition 1.38. Let $(U, u) \in \mathrm{Sm}_{/S_*}$. We define the *henselization* of U at u to be the scheme

$$U_u^h := \lim_{(V,v) \in \mathrm{Nbhd}(U,u)} V.$$

Observe that we have a canonical morphism $p_u \colon U_u^h \to U$.

Remark 1.39. Since the category of affine Nisnevich neighbourhoods is clearly a limit-cofinal subcategory of $\mathrm{Nbhd}(U, u)$, it follows that U_u^h is the affine scheme associated to the henselization $\mathcal{O}_{U,u}^h$ of the local ring $\mathcal{O}_{U,u}$ (see [Sta20, Tag 0BSK]).

Proposition 1.40. *Let $(U, u) \in \mathrm{Sm}_{/S_*}$. Let us denote the given map $U \to S$ by f. Then the diagram*

$$\begin{array}{ccc} \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S}) & \xrightarrow{\quad (-)_u \quad} & \mathcal{S} \\ & {\scriptstyle (f\circ p_u)^*} \searrow \qquad \nearrow {\scriptstyle \Gamma} & \\ & \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/U_u^h}) & \end{array}$$

commutes, where Γ denotes the global sections functor.

Proof: By [Lur09, Proposition 6.2.3.20], it suffices to see that the two functors $f(-)_u$ and $\Gamma \circ (f \circ p_u)^*$ are equivalent when restricted to $\mathrm{Sm}_{/S}$ along the Yoneda embedding. We compute for $X \in \mathrm{Sm}_{/S}$:

$$\begin{aligned} \Gamma(f \circ p_u)^*(h(X)) = \mathrm{map}_{\mathrm{Sm}_{/U_u^h}}(U_u^h, U_u^h \times_S X) &\cong \mathrm{map}_S(U_u^h, X) \\ &\cong \mathrm{map}_S(\lim_{(V,v) \in \mathrm{Nbhd}(U,u)} V, X) \\ &\cong \operatorname*{colim}_{(V,v) \in \mathrm{Nbhd}(U,u)^{\mathrm{op}}} \mathrm{map}_{\mathrm{Sm}_{/S}}(V, X) \\ &\cong h(X)_u, \end{aligned}$$

where the third isomorphy holds because X is finitely presented over S. Furthermore, this isomorphism is clearly natural in X and the claim follows. □

We will now deduce that the ∞-topos $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$ has enough points. We start by recalling the classical statements for sheaves of sets:

Theorem 1.41. *Let S be a noetherian scheme. A morphism $\alpha \colon \mathcal{F} \to \mathcal{G}$ between presheaves of sets on $\mathrm{Sm}_{/S}$ is a Nisnevich-local equivalence if and only if it is a stalkwise equivalence.*

For the proof we will need the following lemma:

Lemma 1.42. *Let $\{U_i \to X\}_{i \in I}$ be a family of étale morphisms in $\mathrm{Sm}_{/S}$ such that, for every field k and any k-point $x \colon \mathrm{Spec}\,(k) \to X$, there is an $i \in I$ and a k-point $x' \colon \mathrm{Spec}\,(k) \to U_i$ lifting x. Assume that S is noetherian. Then there is a finite subset $I_0 \subseteq I$ such that $\{U_i \to X\}_{i \in I_0}$ is a Nisnevich covering family.*

Proof: We use Noetherian induction to assume that the claim holds for all proper closed subschemes $Z \subsetneq X$. Let $\eta \in X$ be a maximal point of X. Then the local ring $O_{X,\eta}$ is an artinian local ring and thus its reduction is given by the residue field $k(\eta)$. By assumption, there is an $\alpha \in I$ and a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(k(\eta)) & \longrightarrow & U_\alpha \\ \downarrow & & \downarrow{\scriptstyle p_\alpha} \\ \operatorname{Spec}\left(O_{X,\eta}\right) & \longrightarrow & X \end{array}$$

Let us write $\mathfrak{m}$ for the maximal ideal of $O_{X,\eta}$. Then $\mathfrak{m}$ is nilpotent as $O^h_{X,\eta}$ is artinian local and thus the canonical morphism

$$\operatorname{Spec}(k(\eta)) \to \operatorname{Spec}\left(O_{X,\eta}\right)$$

is an infinitesimal thickening. Since p_α is étale, so in particular formally étale, there is a lift in the above diagram. As p_α is of finite presentation, the above section extends to an open neighbourhood of η, i.e. there is an open subscheme $V \subseteq X$ containing η and a commutative diagram

$$\begin{array}{ccc} & & U_\alpha \\ & \nearrow & \downarrow{\scriptstyle p_\alpha} \\ V & \hookrightarrow & X \end{array}$$

Let us now consider the complement $X \setminus V$, which we endow with the reduced subscheme structure. By the assumption above, there is a finite $A \subseteq I$ such that the induced morphism

$$\coprod_{i \in A} U_i \times_X (X \setminus V) \to X \setminus V$$

is a Nisnevich covering. Now setting $I_0 = A \cup \{\alpha\}$, we get the claim. □

Proof of Theorem 1.41: One of the implications is taken care of by Lemma 1.35. For the other, Lemma 1.35 shows that we may assume that both $\mathcal{F}$ and $\mathcal{G}$ are Nisnevich sheaves and we have to show that α is an isomorphism if it is a stalkwise isomorphism. So let $U \in \mathrm{Sm}_{/S}$. We would like to show that the morphism

$$\alpha(U) \colon \mathcal{F}(U) \to \mathcal{G}(U)$$

is an isomorphism. We will start by showing injectivity. So let $x, y \in \mathcal{F}(U)$ such that $\alpha(U)(x) = \alpha(U)(y)$. Since α is stalkwise injective, it follows that for every $u \in U$, there is a Nisnevich neighbourhood (T_u, t_u) of u such that

$$x|_{T_u} = y|_{T_u} .$$

Clearly the family $\{T_u \to U\}_{u \in U}$ satisfies the condition of Lemma 1.42 and thus x and y agree on a Nisnevich covering of U. Since $\mathcal{F}$ is a sheaf, it follows that $x = y$.

For surjectivity, let $z \in \mathcal{G}(U)$. Since α is stalkwise surjective, using Lemma 1.42 we see that there is a Nisnevich covering $\{T_i \to U\}_{i \in I}$ and an $x_i \in \mathcal{F}(T_i)$ such that

$$\alpha(T_i)(x_i) = z|_{T_i} .$$

Using the injectivity we have shown above, we see that for every $i, j \in I$

$$x_i|_{T_i \times_U T_j} = x_j\big|_{T_i \times_U T_j} .$$

Thus, by the sheaf condition, there is an $x \in \mathcal{F}(U)$ such that

$$x|_{T_i} = x_i$$

and thus $\alpha(U)(x) = z$, since $\mathcal{G}$ is a sheaf. □

We will now demonstrate how to upgrade the above result to general sheaves of spaces. For this, we need the following construction:

Construction 1.43. Let $\mathcal{F} \in \mathrm{Psh}(\mathrm{Sm}_{/S})$ and let $X \in \mathrm{Sm}_{/S}$. For $x \in \mathcal{F}(X)$, we define the presheaf

$$\begin{aligned} \pi_n^X(\mathcal{F}, x) \colon \ \mathrm{Sm}_{/X} &\to \mathrm{Set} \\ U &\mapsto \pi_n(\mathcal{F}(U), x|_U). \end{aligned}$$

Lemma 1.44. *Let $n \in \mathbb{N}$ and let $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})_{\leq n}$ be an n-truncated sheaf. Let $X \in \mathrm{Sm}_{/S}$ and $x \in \mathcal{F}(X)$. Then the presheaf $\pi_n^X(\mathcal{F}, x)$ is a Nisnevich sheaf on* $\mathrm{Sm}_{/X}$.

Proof: Let

$$\begin{array}{ccc} U \times_T V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & T \end{array}$$

be a Nisnevich square in $\mathrm{Sm}_{/X}$. Since $\mathcal{F}$ is a Nisnevich sheaf, the induced square

$$\begin{array}{ccc} \mathcal{F}(T) & \longrightarrow & \mathcal{F}(V) \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U \times_T V) \end{array}$$

of n-truncated spaces is a pullback square in $\mathcal{S}$. We thus get an induced exact sequence of homotopy groups

$$0 \to \pi_n(\mathcal{F}(T), x|_T) \to \pi_n(\mathcal{F}(U), x|_U) \times \pi_n(\mathcal{F}(V), x|_V) \to \pi_n(\mathcal{F}(U \times_T V), x|_{U \times_T V})$$

which precisely says that the induced square

$$\begin{array}{ccc} \pi_n(\mathcal{F}(T), x|_T) & \longrightarrow & \pi_n(\mathcal{F}(V), x|_V) \\ \downarrow & & \downarrow \\ \pi_n(\mathcal{F}(U), x|_U) & \longrightarrow & \pi_n(\mathcal{F}(U \times_X V), x|_{U \times_T V}) \end{array}$$

is a pullback. Since clearly $\pi_n^X(\mathcal{F}, x)(\emptyset) = *$, the claim follows by Theorem 1.15. □

Proposition 1.45. *Let S be a noetherian scheme of finite Krull dimension. Let $n \in \mathbb{N}$ and let $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})_{\leq n}$ be an n-truncated presheaf such that for every $(U, u) \in \mathrm{Sm}_{/S*}$ the stalk $\mathcal{F}_u$ is contractible. Then $\mathcal{F}$ is equivalent to the terminal object of $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$.*

Proof: We argue by induction on n. If $n = 0$, the claim follows from Theorem 1.41. So let $n \geq 1$. Let $X \in \mathrm{Sm}_{/S}$ and let $p \in X$ be a point. Since homotopy groups commute with filtered colimits, the canonical morphism

$$\pi_n^X(\mathcal{F}, x)_p \to \pi_n(\mathcal{F}_p, x_p)$$

is an isomorphism for every $x \in \mathcal{F}(X)$. But, by assumption, we have $\pi_n(\mathcal{F}_p, x_p) \cong *$ and, by Lemma 1.44, the presheaf $\pi_n^X(\mathcal{F}, x)$ is a sheaf of sets. Thus, by Theorem 1.41, it follows that $\pi_n^X(\mathcal{F}, x) \simeq *$ and in particular $\pi_n(\mathcal{F}(X), x) = *$. Therefore, the space $\mathcal{F}(X)$ is $(n-1)$-truncated for every $X \in \mathrm{Sm}_{/S}$ and thus $\mathcal{F}$ is $(n-1)$-truncated. So the claim follows by the induction hypothesis. □

Theorem 1.46. *Let S be a noetherian scheme of finite Krull dimension. Let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathrm{Psh}(\mathrm{Sm}_{/S})$. Then α is a Nisnevich-local equivalence if and only if α is a stalkwise equivalence.*

Proof: Again, one implication is just Lemma 1.35. For the converse, we may assume that $\mathcal{F}$ and $\mathcal{G}$ are Nisnevich sheaves by Lemma 1.35 and we would like to show that α is an equivalence. We consider, for every $n \in \mathbb{N}$, the truncation functor

$$\tau_{\leq n}\colon\ \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S}) \to \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})_{\leq n}.$$

By [Lur09, Proposition 5.5.6.28], this functor is given by the composite

$$\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S}) \hookrightarrow \mathrm{Psh}(\mathrm{Sm}_{/S}) \xrightarrow{\tau_{\leq n}^{\mathrm{obj}}} \mathrm{Psh}(\mathrm{Sm}_{/S})_{\leq n} \to\ \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})_{\leq n}$$

where $\tau_{\leq n}^{\mathrm{obj}}$ denotes the objectwise truncation functor and the last arrow is given by the restriction of the localization functor $L\colon \mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$. Since objectwise truncation preserves colimits, it again follows from Lemma 1.35 that, for every $n \in \mathbb{N}$, the induced morphism

$$\tau_{\leq n}\alpha\colon \tau_{\leq n}\mathcal{F} \to \tau_{\leq n}\mathcal{G}$$

is a stalkwise equivalence. Since, by Corollary 1.29, Postnikov towers converge in the ∞-topos $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$, it follows that α is an equivalence if $\tau_{\leq n}\alpha$ is an equivalence for all n. So we have reduced the situation to the case that $\mathcal{F}$ and $\mathcal{G}$ are n-truncated for some $n \in \mathbb{N}$. Now, for every $f\colon X \to S \in \mathrm{Sm}_{/S}$ and $x \in \mathcal{G}(X)$, consider the fiber $H \in \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/X})$ defined by the cartesian square

$$\begin{array}{ccc} H & \longrightarrow & f^*\mathcal{F} \\ \big\downarrow & & \big\downarrow{\scriptstyle f^*\alpha} \\ * & \xrightarrow{\ x\ } & f^*\mathcal{G} \end{array}$$

in $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/X})$. Since n-truncated objects are stable under finite limits, it follows that H is n-truncated. Then, since the stalk functor preserves finite limits, it follows that, for every $Y \to X$ in $\mathrm{Sm}_{/X}$ and $u \in Y$, the space H_u is contractible, because $f^*\alpha$ is a stalkwise equivalence. Thus, by Proposition 1.45, it follows that $H \simeq *$. Now, applying the global sections functor shows that, for every $x \in \mathcal{G}(X)$, the homotopy fiber of

$$\alpha(X) \colon \mathcal{F}(X) \to \mathcal{G}(X)$$

over x is contractible. In other words, the morphism $\alpha(X)$ is an equivalence and the theorem follows. □

Remark 1.47. Let S be a noetherian scheme. Let $U \in S^{\mathrm{Nis}}$ and let $u \in U$ be a point. Observe that the category $\mathrm{Nbhd}(U, u)$ defined in Definition 1.32 is in fact a subcategory of the slice S^{Nis}/U and one thus can also define a stalk functor

$$(-)_u \colon \mathrm{Psh}(U^{\mathrm{Nis}}) \to \mathcal{S}.$$

Observe that one also gets analogous versions of the results above for these stalk functors.

We will use the opportunity to pay our debts from the last subsection:

Proof of Lemma 1.20: Let $n \geq 0$ and let $\gamma \colon \mathcal{F} \to h(X)$ be an n-connective morphism in S_{Nis}. We will inductively show that γ is weakly n-connective. We start with $n = 0$. Let $f \colon U \to X$ be in S^{Nis} and let $u \in U$ be any point. By assumption, the map γ is an effective epimorphism and since the stalk functor

$$(-)_u \colon S_{\mathrm{Nis}} \to \mathcal{S}$$

preserves colimits and finite limits, it follows that

$$\gamma_u \colon \mathcal{F}_u \to h(X)_u$$

is an effective epimorphism. It follows that there is a Nisnevich neighbourhood $(V, v) \to (U, u)$ and a morphism $\phi \colon h(V) \to \mathcal{F}$ such that $\gamma(\phi)$ and $f \colon U \to X$ represent the same element in the stalk $h(X)_u$. This precisely means that there is a Nisnevich neighbourhood (V', v') with maps $g \colon (V', v') \to (U, u)$ and $g' \colon (V', v') \to (V, v)$ such that the diagram

$$\begin{array}{ccccc} & & h(V) & \xrightarrow{\phi} & \mathcal{F} \\ & \overset{g'}{\nearrow} & & & \downarrow{\scriptstyle\gamma} \\ h(V') & \xrightarrow{g} & h(U) & \xhookrightarrow{f} & h(X) \end{array}$$

commutes. In particular, the space $\mathcal{F}_{gf}(V')$ is non-empty, so condition iii) a) of Definition 1.19 is satisfied.

Now, for $n > 0$, if γ is n-connective, it is in particular 0-connective and thus the above argument shows that condition iii) a) is again satisfied. So let $a \colon h(U) \to \mathcal{F}$ and $b \colon h(V) \to \mathcal{F}$ be arbitrary morphisms. Since γ is n-connective, we know that the diagonal

$$\Delta \colon \mathcal{F} \to \mathcal{F} \times_{h(X)} \mathcal{F}$$

is $(n-1)$-connective. We now consider the diagram

$$\begin{array}{ccccc} h(U) & \xrightarrow{\varphi} & h(U)\times_{h(X)}\mathcal{F} & \xrightarrow{\mathrm{pr}_1} & h(U) \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle (a,\mathrm{id})} & & \downarrow{\scriptstyle a} \\ \mathcal{F} & \xrightarrow{\Delta} & \mathcal{F}\times_{h(X)}\mathcal{F} & \xrightarrow{\mathrm{pr}_1} & \mathcal{F} \end{array}$$

where $\varphi = \mathrm{id}\times a$ and the right square and the outer square are both pullbacks. Hence the left square is a pullback. Since k-connective maps in any ∞-topos are stable under pullbacks (see [Lur09, Proposition 6.5.1.16]), we know that φ is $(n-1)$- connective. Now we consider

$$\begin{array}{ccccc} h(U)\times_{\mathcal{F}} h(V) & \xrightarrow{\theta} & h(U)\times_{h(X)} h(V) & \xrightarrow{\mathrm{pr}_2} & h(V) \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle (\mathrm{id},b)} & & \downarrow{\scriptstyle b} \\ h(U) & \xrightarrow{\varphi} & h(U)\times_{h(X)}\mathcal{F} & \xrightarrow{\mathrm{pr}_2} & \mathcal{F} \end{array}$$

where again all squares are pullback squares. Thus θ is $(n-1)$-connective. By induction, we get that θ is weakly $(n-1)$-connective and hence γ is weakly n-connective, as desired. □

2. Unstable Motivic Homotopy Theory

In this chapter we will construct and study the unstable motivic homotopy category of a quasi-compact and quasi-separated scheme S. In section 2.1 we will define the unstable motivic homotopy category as the full subcategory of $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$ spanned by the $\mathbb{A}^1$-invariant sheaves. In particular, it is a reflective subcategory of $\mathrm{Psh}(\mathrm{Sm}_{/S})$ and the goal of section 2.2 is to study the associated localization functor. In section 2.3 we will study the functoriality properties of the assignment $S \mapsto \mathcal{H}(S)$ and deduce the smooth base change (Proposition 2.24) and smooth projection formula (Proposition 2.26). The goal of section 2.4 is to better understand the properties of the motivic direct image functor $i_*^{\mathcal{H}}$ along a closed immersion i and show that it preserves weakly contractible colimits (Proposition 2.27). This will be an important input for the proof of Theorem 2.33, which is the main objective of section 2.5. This section is mostly inspired by [Hoy17] and [Kha16].

2.1. The Unstable Motivic Homotopy Category

Notation 2.1. As usual, we will write $\mathbb{A}^1 = \mathrm{Spec}\,(\mathbb{Z}[T])$ for the affine line. For a scheme S, we will write

$$\mathbb{A}^1_S = \mathbb{A}^1 \times_{\mathrm{Spec}(\mathbb{Z})} S$$

for the affine line over S. However, we will often drop the index S when there is no danger of confusion.

Definition 2.2. Let S be a quasi-compact and quasi-separated scheme. Let W be the class of all morphisms of the form

$$X \times \mathbb{A}^1 \to X$$

for $X \in \mathrm{Sm}_{/S}$. We define the *unstable motivic homotopy category* $\mathcal{H}(S)$ to be the full subcategory of $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$ consisting of the W-local objects. By [Lur09, Proposition 5.5.4.15] and since $\mathrm{Sm}_{/S}$ is essentially small, we get a left adjoint

$$L\colon \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S}) \to \mathcal{H}(S)$$

of the inclusion $i\colon \mathcal{H}(S) \hookrightarrow \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$, which exhibits $\mathcal{H}(S)$ as the reflective localization obtained by inverting all morphisms of the form $\mathrm{pr}_1\colon h(X) \times h(\mathbb{A}^1) \to h(X)$. Combining this with the localization functor of Definition 1.7, we get a left adjoint

$$L_S^{\mathrm{mot}}\colon \mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathcal{H}(S)$$

of the inclusion $j\colon \mathcal{H}(S) \hookrightarrow \mathrm{Psh}(\mathrm{Sm}_{/S})$. We call a morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ a *motivic equivalence* if $L_S^{\mathrm{mot}}(\alpha)$ is an equivalence. We call an object $\mathcal{F} \in \mathcal{H}(S)$ a *motivic space*.

Remark 2.3. When spelling out the above definition explicitly, we see that a presheaf $\mathcal{F} \in \mathrm{Psh}(\mathrm{Sm}_{/S})$ is a motivic space if and only if it is a Nisnevich sheaf and for every $X \in \mathrm{Sm}_{/S}$, the morphism

$$\mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1)$$

induced by the projection $X \times \mathbb{A}^1 \to X$ is an equivalence.

Definition 2.4. We will write $\mathrm{Psh}^{\mathbb{A}^1}(\mathrm{Sm}_{/S})$ for the full subcategory of $\mathrm{Psh}(\mathrm{Sm}_{/S})$ spanned by all presheaves $\mathcal{F}$ such that for every $X \in \mathrm{Sm}_{/S}$ the morphism

$$\mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1)$$

induced by the projection $X \times \mathbb{A}^1 \to X$ is an equivalence. Again, by [Lur09, Proposition 5.5.4.15], the inclusion $k\colon \mathrm{Psh}^{\mathbb{A}^1}(\mathrm{Sm}_{/S}) \hookrightarrow \mathrm{Psh}(\mathrm{Sm}_{/S})$ has a left adjoint

$$L_S^{\mathbb{A}^1}\colon \mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathrm{Psh}^{\mathbb{A}^1}(\mathrm{Sm}_{/S}),$$

which exhibits $\mathrm{Psh}^{\mathbb{A}^1}(\mathrm{Sm}_{/S})$ as the reflective localization obtained by inverting all morphisms of the form $\mathrm{pr}_1\colon h(X) \times h(\mathbb{A}^1) \to h(X)$. We call a morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ in $\mathrm{Psh}(\mathrm{Sm}_{/S})$ an $\mathbb{A}^1$-*local equivalence* if $L_S^{\mathbb{A}^1}(\alpha)$ is an equivalence.

Notation 2.5. From now on, we will identify $\mathrm{Sm}_{/S}$ with the full subcategory of $\mathrm{Psh}(\mathrm{Sm}_{/S})$ spanned by the representable objects and omit the $h(-)$ from the notation whenever it is convenient.

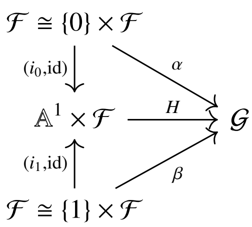
Definition 2.6. Let S be a quasi-compact and quasi-separated scheme. Let $i_0\colon S \to \mathbb{A}^1$ denote the zero section and $i_1\colon S \to \mathbb{A}^1$ the section at one. Let

$$\alpha, \beta\colon \mathcal{F} \to \mathcal{G}$$

be two morphism in $\mathrm{Psh}(\mathrm{Sm}_{/S})$. We will say that α and β are $\mathbb{A}^1$-*homotopic*, if there is a morphism

$$H\colon \mathbb{A}^1 \times \mathcal{F} \to \mathcal{G}$$

in $\mathrm{Psh}(\mathrm{Sm}_{/S})$ such that the diagram



commutes. We call H an $\mathbb{A}^1$-*homotopy* from α to β.

We immediately see the following:

Lemma 2.7. *Let $\alpha, \beta \colon \mathcal{F} \to \mathcal{G}$ be two $\mathbb{A}^1$-homotopic maps in* $\mathrm{Psh}(\mathrm{Sm}_{/S})$. *Then $L_S^{\mathbb{A}^1}(\alpha)$ and $L_S^{\mathbb{A}^1}(\beta)$ are equivalent in* $\mathrm{Psh}^{\mathbb{A}^1}(\mathrm{Sm}_{/S})$.

2.2. Motivic Localization

The goal of this section is to give a more explicit description of the localization functor $L_S^{\mathrm{mot}} \colon \mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathcal{H}(S)$, which will then allow us to deduce many of its useful properties.

Construction 2.8. Consider the functor

$$\Delta^\bullet \colon \mathbf{\Delta} \to \mathrm{Sm}_{/\,\mathrm{Spec}(\mathbb{Z})},$$

which associates to the partially ordered set $[n]$ the smooth scheme

$$\Delta^n := \mathrm{Spec}\,(\mathbb{Z}[T_0, \dots T_n]/T_0 + \dots + T_n = 1))$$

and sends a morphism $\alpha \colon [n] \to [k]$ to the morphism of schemes

$$\Delta^\alpha \colon \Delta^n \to \Delta^k,$$

whose corresponding map of rings is determined by

$$T_i \mapsto \Sigma_{a \in \alpha^{-1}(i)} T_a.$$

If S is any quasi-compact quasi-separated scheme, base changing along the canonical morphism $S \to \mathrm{Spec}\,(\mathbb{Z})$ gives us a functor

$$\Delta_S^\bullet \colon \mathbf{\Delta} \to \mathrm{Sm}_{/S}\ .$$

Again, we will often drop the index S.

Definition 2.9. We define Sing_S to be the functor given by the composite

$$\mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Psh}(\mathrm{Sm}_{/S})) \xrightarrow{|-|} \mathrm{Psh}(\mathrm{Sm}_{/S}),$$

where the first functor takes a presheaf $\mathcal{F}$ to the simplicial object $\mathcal{F}(- \times \Delta_S^\bullet)$.

Lemma 2.10. *Let $f, g\colon \mathcal{F} \to \mathcal{G}$ be two $\mathbb{A}^1$-homotopic morphisms in* $\mathrm{Psh}(\mathrm{Sm}_{/S})$*. Then* $\mathrm{Sing}_S(f)$ *and* $\mathrm{Sing}_S(g)$ *are equivalent in* $\mathrm{Psh}(\mathrm{Sm}_{/S})$*.*

Proof: First of all we note that for every $n \in \mathbb{N}$, there is an isomorphism of rings

$$\begin{aligned} \mathbb{Z}[T_0, \ldots, T_n]/(T_0 + \ldots + T_n = 1) &\to \mathbb{Z}[T_0, \ldots, T_{n-1}] \\ T_i &\mapsto \begin{cases} T_i & i < n \\ 1 - \sum_i^{n-1} T_i & i = n \end{cases} \end{aligned}$$

inducing an isomorphism $\mathbb{A}^n_S \to \Delta^n_S$. Furthermore this isomorphism makes the diagram

$$\begin{array}{ccc} S & \xrightarrow{i_0} & \mathbb{A}^1_S \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \Delta^0_S & \xrightarrow{\Delta^{\delta_0}_S} & \Delta^1_S \end{array}$$

commute, where i_0 is the zero section and $\delta_0\colon [0] \hookrightarrow [1]$ the inclusion onto 1. We get an analogous commutative diagram for the section at one i_1 and δ_1. Thus the assumptions provide a commutative diagram

$$\begin{array}{ccc} \Delta^0_S \times \mathcal{F} & & \\ {\scriptstyle \Delta^{\delta_0}_S \times \mathrm{id}}\downarrow & \searrow^{f} & \\ \Delta^1_S \times \mathcal{F} & \xrightarrow{H} & \mathcal{G} \\ {\scriptstyle \Delta^{\delta_1}_S \times \mathrm{id}}\uparrow & \nearrow_{g} & \\ \Delta^0_S \times \mathcal{F} & & \end{array}$$

and since Sing_S commutes with products as a sifted colimit of product preserving functors, it follows that it suffices to see that the morphisms

$$\mathrm{Sing}_S(\Delta^{\delta_0}_S), \mathrm{Sing}_S(\Delta^{\delta_1}_S)\colon * = \mathrm{Sing}_S(\Delta^0_S) \to \mathrm{Sing}_S(\Delta^1_S)$$

agree in $\mathrm{Psh}(\mathrm{Sm}_{/S})$. For this we consider for $n \in \mathbb{N}$ the simplicial object

$$\begin{aligned} \Delta^n\colon \mathbf{\Delta}^{\mathrm{op}} &\to \mathrm{Psh}(\mathrm{Sm}_{/S}) \\ [k] &\mapsto \coprod_{t \in \Delta^n_k} * \end{aligned}$$

and observe that this construction assembles to a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Psh}(\mathrm{Sm}_{/S})$. Now the Yoneda lemma provides us with a natural isomorphism

$$\mathrm{map}_{\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Psh}(\mathrm{Sm}_{/S}))}(\Delta^n, h(\Delta^1_S)(- \times \Delta^\bullet_S)) \cong \mathrm{map}_{\mathrm{Sm}_{/S}}(\Delta^n_S, \Delta^1_S)$$

and thus we may consider the unique morphism of simplicial objects $K\colon \Delta^1 \to h(\Delta^1_S)(- \times \Delta^\bullet_S)$ corresponding to the identity under the above isomorphism. By the naturality we thus get a commutative diagram

$$\begin{array}{ccc} \Delta^0 & & \\ {\scriptstyle \delta_0}\downarrow & \searrow^{\alpha} & \\ \Delta^1 & \xrightarrow{K} & h(\Delta^1_S)(- \times \Delta^\bullet_S) \\ {\scriptstyle \delta_1}\uparrow & \nearrow_{\beta} & \\ \Delta^0 & & \end{array}$$

and under the identification $\Delta^0 = h(\Delta^0_S)(- \times \Delta^\bullet_S)$ the map α is induced by the morphism $\Delta^{\delta_0}_S : \Delta^0_S \to \Delta^1_S$ and β is induced by $\Delta^{\delta_1}_S : \Delta^0_S \to \Delta^1_S$. Thus applying the geometric realization functor gives a commutative diagram

$$\begin{array}{ccc} * = \mathrm{Sing}_S(\Delta^0_S) & & \\ \downarrow & \searrow^{\mathrm{Sing}_S(\Delta^{\delta_0}_S)} & \\ |\Delta^1| & \xrightarrow{|K|} & \mathrm{Sing}_S(\Delta^1_S) \\ \uparrow & \nearrow_{\mathrm{Sing}_S(\Delta^{\delta_1}_S)} & \\ * = \mathrm{Sing}_S(\Delta^0_S) & & \end{array}$$

and the claim follows since $|\Delta^1| \simeq *$ in $\mathrm{Psh}(\mathrm{Sm}_{/S})$. □

Proposition 2.11. *Let $\mathcal{F} \in \mathrm{Psh}(\mathrm{Sm}_{/S})$ be a presheaf. Then $\mathrm{Sing}_S(\mathcal{F})$ is $\mathbb{A}^1$-invariant. Furthermore, the functor*

$$\mathrm{Sing}_S : \mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathrm{Psh}^{\mathbb{A}^1}(\mathrm{Sm}_{/S})$$

is left adjoint to the inclusion functor $i\colon \mathrm{Psh}^{\mathbb{A}^1}(\mathrm{Sm}_{/S}) \hookrightarrow \mathrm{Psh}(\mathrm{Sm}_{/S})$

Proof: We would like to see that the map

$$f\colon \mathrm{Sing}_S(\mathcal{F}) \to \mathrm{Sing}_S(\mathcal{F}(- \times \mathbb{A}^1_S))$$

induced by the unique morphism $\mathbb{A}^1_S \to S$ is an equivalence. We consider the map

$$\gamma\colon \mathcal{F}(- \times \mathbb{A}^1_S) \to \mathcal{F}$$

induced by the zero section $i_0\colon S \to \mathbb{A}^1_S$ and note that $\mathrm{Sing}_S(\gamma)$ is a left inverse of f. By Lemma 2.10 it now suffices to see that the composite

$$\mathcal{F}(- \times \mathbb{A}^1_S) \to \mathcal{F} \to \mathcal{F}(- \times \mathbb{A}^1_S)$$

is $\mathbb{A}^1$-homotopic to the identity. For this we consider the morphism

$$H\colon \mathbb{A}^1_S \times \mathcal{F}(- \times \mathbb{A}^1_S) \to \mathbb{A}^1_S$$

corresponding under adjunction to the morphism

$$\mathcal{F}(- \times \mathbb{A}^1_S) \to \mathcal{F}(- \times \mathbb{A}^1_S \times \mathbb{A}^1_S)$$

induced by the multiplication map

$$\begin{aligned} m\colon \mathbb{A}^1_S \times \mathbb{A}^1_S &\to \mathbb{A}^1_S \\ (x, y) &\mapsto xy \end{aligned}$$

which provides the desired $\mathbb{A}^1$-homotopy.

To show that Sing_S is left adjoint to the inclusion, we observe that we have a canonical natural transformation

$$\eta\colon \mathrm{id}_{\mathrm{Psh}(\mathrm{Sm}_{/S})} \to i \circ \mathrm{Sing}_S$$

and we have to show that, for any $\mathcal{G} \in \mathrm{Psh}^{\mathbb{A}^1}(\mathrm{Sm}_{/S})$ and $\mathcal{F} \in \mathrm{Psh}(\mathrm{Sm}_{/S})$, the induced morphism

$$\mathrm{map}_{\mathrm{Psh}(\mathrm{Sm}_{/S})}(i\,\mathrm{Sing}_S(\mathcal{F}), i\mathcal{G}) \xrightarrow{\eta^*_{\mathcal{F}}} \mathrm{map}_{\mathrm{Psh}(\mathrm{Sm}_{/S})}(\mathcal{F}, i\mathcal{G})$$

is an equivalence. Since

$$\mathrm{map}_{\mathrm{Psh}(\mathrm{Sm}_{/S})}(i\,\mathrm{Sing}_S(\mathcal{F}), i\mathcal{G}) \simeq \lim_{[n]\in\Delta} \mathrm{map}_{\mathrm{Psh}(\mathrm{Sm}_{/S})}(\mathcal{F}(- \times \Delta^n_S), \mathcal{G}),$$

it suffices to see that the canonical morphism

$$\mathcal{F} \to \mathcal{F}(- \times \Delta^n_S)$$

induced by the projection $\Delta^n_S \to S$ is an $\mathbb{A}^1$-local equivalence. We have seen above that $\Delta^n_S \cong \mathbb{A}^n_S$ and since $\mathbb{A}^n_S \cong \mathbb{A}^{n-1}_S \times \mathbb{A}^1_S$ we may reduce to showing that the map

$$\beta\colon \mathcal{F} \to \mathcal{F}(- \times \mathbb{A}^1_S)$$

induced by $\mathbb{A}^1_S \to S$ is an equivalence. But the map γ from the beginning of the proof provides a left inverse of β and the $\mathbb{A}^1$-homotopy K from above shows that $\beta \circ \gamma$ is $\mathbb{A}^1$-homotopic to the identity which completes the proof. □

Remark 2.12. Note that, since the inclusion $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S}) \to \mathrm{Psh}(\mathrm{Sm}_{/S})$ does not preserve geometric realizations of simplicial objects in general, the sheaf $\mathrm{Sing}_S(\mathcal{F})$ is not necessarily a Nisnevich sheaf if $\mathcal{F}$ is.

We recall the following definition from [GK17, §1.2]:

Definition 2.13. Let $\mathcal{D}$ be a presentable ∞-category and let $\mathcal{C} \subseteq \mathcal{D}$ be a reflective subcategory with localization functor $L\colon \mathcal{D} \to \mathcal{C}$. Then L is called *locally cartesian* if, for any diagram $A \to B \leftarrow C$ where $A, B \in \mathcal{C}$ and $C \in \mathcal{D}$, the canonical morphism

$$L(A \times_B C) \to L(A) \times_{L(B)} L(C) \simeq A \times_B L(C)$$

is an equivalence.

One immediately verifies the following:

Lemma 2.14. *Let $\mathcal{D}$ be a presentable ∞-category with universal colimits and let $\mathcal{C}$ be a reflective subcategory with localization functor $L\colon \mathcal{D} \to \mathcal{C}$. If L is locally cartesian, then colimits are universal in $\mathcal{C}$.*

Corollary 2.15. *The functor $L_S^{\mathbb{A}^1}\colon \mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathrm{Psh}^{\mathbb{A}^1}(\mathrm{Sm}_{/S})$ is locally cartesian. In particular, colimits are universal in $\mathrm{Psh}^{\mathbb{A}^1}(\mathrm{Sm}_{/S})$.*

Proof: By Lemma 2.14 and Proposition 2.11, it suffices to see that, for any diagram $A \to B \leftarrow C$ in $\mathrm{Psh}(\mathrm{Sm}_{/S})$ where A and B are $\mathbb{A}^1$-invariant, the canonical morphism

$$\mathrm{Sing}_S(A \times_B C) \to \mathrm{Sing}_S(A) \times_{\mathrm{Sing}_S(B)} \mathrm{Sing}_S(C) \simeq A \times_B \mathrm{Sing}_S(C)$$

is an equivalence. Observe that, since A and B are $\mathbb{A}^1$-invariant, the canonical morphism

$$(A \times_B C)(- \times \Delta_S^\bullet) \to A(- \times \Delta_S^\bullet) \times_{B(- \times \Delta_S^\bullet)} C(- \times \Delta_S^\bullet) \simeq \underline{A} \times_{\underline{B}} C(- \times \Delta_S^\bullet)$$

of simplicial objects in $\mathrm{Psh}(\mathrm{Sm}_{/S})$ is an equivalence. Here $\underline{A}$ and $\underline{B}$ denote the constant simplicial objects at A and B. Hence it follows, since colimits are universal in $\mathrm{Psh}(\mathrm{Sm}_{/S})$, that the canonical morphism

$$\begin{aligned}\mathrm{Sing}_S(A \times_B C) \simeq |(A \times_B C)(- \times \Delta_S^\bullet)| \to & |\underline{A} \times_{\underline{B}} C(- \times \Delta_S^\bullet)| \\ \simeq & A \times_B |C(- \times \Delta_S^\bullet)| \\ \simeq & A \times_B \mathrm{Sing}_S(C)\end{aligned}$$

is an equivalence. □

2.16. By abuse of notation, we will also write $L_S^{\mathrm{mot}}\colon \mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathrm{Psh}(\mathrm{Sm}_{/S})$ for the composition of the localization functor

$$L_S^{\mathrm{mot}}\colon \mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathcal{H}(S)$$

with the inclusion $\mathcal{H}(S) \hookrightarrow \mathrm{Psh}(\mathrm{Sm}_{/S})$. Similarly we will write

$$L_S^{\mathrm{Nis}}\colon \mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathrm{Psh}(\mathrm{Sm}_{/S}) \qquad \text{and} \qquad \mathrm{Sing}_S\colon \mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathrm{Psh}(\mathrm{Sm}_{/S})$$

for the compositions of the localization functors with the corresponding inclusions. Then the above adjunctions yield natural transformations

$$\mathrm{id}_{\mathrm{Psh}(\mathrm{Sm}_{/S})} \to L_S^{\mathrm{Nis}} \qquad \text{and} \qquad \mathrm{id}_{\mathrm{Psh}(\mathrm{Sm}_{/S})} \to \mathrm{Sing}_S$$

and thus we also get a natural transformation

$$\mathrm{id}_{\mathrm{Psh}(\mathrm{Sm}_{/S})} \to L_S^{\mathrm{Nis}} \circ \mathrm{Sing}_S .$$

Proposition 2.17. *The localization functor $L_S^{\mathrm{mot}}\colon \mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathcal{H}(S)$ is equivalent to the transfinite composition*

$$\Phi := \operatorname*{colim}_{n \in \mathbb{N}} (L_S^{\mathrm{Nis}} \circ \mathrm{Sing}_S)^n.$$

Proof: First of all, note that, for any $\mathcal{F} \in \mathrm{Psh}(\mathrm{Sm}_{/S})$, the presheaf $\Phi(\mathcal{F})$ is a filtered colimit of Nisnevich sheaves. Since finite limits commute with filtered colimits, it follows that a filtered colimit of presheaves which satisfy Nisnevich excision satisfies Nisnevich excision as well. So $\Phi(\mathcal{F})$ is a Nisnevich sheaf by Theorem 1.15. Furthermore, observe that one also gets an equivalence

$$\operatorname*{colim}_{n\in\mathbb{N}}(\mathrm{Sing}_S \circ L_S^{\mathrm{Nis}})^n(\mathrm{Sing}_S(\mathcal{F})) \simeq \Phi(\mathcal{F}),$$

which can be seen through an easy cofinality argument. It follows that $\mathcal{F}$, as a filtered colimit of $\mathbb{A}^1$-invariant presheaves, is $\mathbb{A}^1$-invariant. It remains to show that the induced morphism $\eta_{\mathcal{F}} : \mathcal{F} \to \Phi(\mathcal{F})$ induces an equivalence

$$\mathrm{map}_{\mathrm{Psh}(\mathrm{Sm}_{/S})}(\Phi(\mathcal{F}), \mathcal{G}) \xrightarrow{\simeq} \mathrm{map}_{\mathrm{Psh}(\mathrm{Sm}_{/S})}(\mathcal{F}, \mathcal{G}).$$

for $\mathcal{G}$ in $\mathcal{H}(S)$. By pulling out the colimit and by induction, it suffices to show that the induced map $\mathcal{F} \to L_S^{\mathrm{Nis}} \circ \mathrm{Sing}(\mathcal{F})$ induces an equivalence

$$\mathrm{map}_{\mathrm{Psh}(\mathrm{Sm}_{/S})}(L_S^{\mathrm{Nis}} \circ \mathrm{Sing}(\mathcal{F}), \mathcal{G}) \xrightarrow{\simeq} \mathrm{map}_{\mathrm{Psh}(\mathrm{Sm}_{/S})}(\mathcal{F}, \mathcal{G}).$$

But this follows by definition of L_S^{Nis} and Proposition 2.11. Hence Φ is a left adjoint of the inclusion $\mathcal{H}(S) \to \mathrm{Psh}(\mathrm{Sm}_{/S})$ and the claim follows by the uniqueness of adjoints. □

Corollary 2.18. *The localization functor* $L_S^{\mathrm{mot}} \colon \mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathcal{H}(S)$ *preserves products.*

Proof: This follows from the description above: The functor L_S^{Nis} commutes with products (it is even left exact) and Sing_S preserves products since it is a sifted colimit of product preserving functors. Thus L_S^{mot}, as a filtered colimit of product preserving functors, commutes with products. □

Remark 2.19. Note that the functor L_S^{Nis} is clearly locally cartesian as it preserves finite limits. It follows that the localization functor $L_S^{\mathrm{mot}} \colon \mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathrm{Psh}(\mathrm{Sm}_{/S})$ is locally cartesian as well. Hence colimits are universal in $\mathcal{H}(S)$.

We will also make the following observation for later use:

Lemma 2.20. *Every object in* $\mathcal{H}(S)$ *can be written as a sifted colimit of objects of the form* $L_S^{\mathrm{mot}}(T)$ *for some* $T \in \mathrm{Sm}_{/S}$.

Proof: Observe that any Nisnevich sheaf $\mathcal{F} \colon \mathrm{Sm}_{/S}^{\mathrm{op}} \to \mathcal{S}$ is in particular a product preserving functor. It follows from [Lur09, Proposition 5.5.8.22] that $\mathcal{F}$ can be written as a sifted colimit of representables in $\mathrm{Psh}(\mathrm{Sm}_{/S})$. Now, applying the motivic localization functor L_S^{mot} gives the claim. □

2.3. Functoriality

In this section we will analyse the functoriality of the assignment

$$S \mapsto \mathcal{H}(S).$$

Construction 2.21. Let $f\colon X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. This induces a functor

$$- \times_Y X\colon \mathrm{Sm}_{/Y} \to \mathrm{Sm}_{/X}$$

given by pulling back along f, which is a morphism of sites. We thus get an adjunction

$$f^*\colon \mathrm{Psh}(\mathrm{Sm}_{/Y}) \rightleftarrows \mathrm{Psh}(\mathrm{Sm}_{/X}) : f_*,$$

where the right adjoint f_* is given by precomposing with the pullback functor and f^* is given by left Kan-extension. In particular, the functor f^* preserves colimits and makes the diagram

$$\begin{array}{ccc} \mathrm{Psh}(\mathrm{Sm}_{/Y}) & \xrightarrow{f^*} & \mathrm{Psh}(\mathrm{Sm}_{/X}) \\ \uparrow & & \uparrow \\ \mathrm{Sm}_{/Y} & \xrightarrow{(- \times_Y X)} & \mathrm{Sm}_{/X} \end{array}$$

commute, where the vertical arrows are given by the Yoneda embedding. We have seen in Remark A.18 that f^* preserves Nisnevich-local equivalences. Furthermore, the above diagram shows that f^* preserves $\mathbb{A}^1$-projections when restricted to $\mathrm{Sm}_{/Y}$. So f^* preserves motivic equivalences and it follows by adjunction that f_* preserves motivic spaces. In other words, it restricts to a functor

$$f_*^{\mathcal{H}}\colon \mathcal{H}(X) \to \mathcal{H}(Y).$$

We also see that $f_*^{\mathcal{H}}$ has a left adjoint

$$f^*_{\mathcal{H}}\colon \mathcal{H}(Y) \to \mathcal{H}(X)$$

given by the composition

$$\mathcal{H}(Y) \hookrightarrow \mathrm{Psh}(\mathrm{Sm}_{/Y}) \xrightarrow{f^*} \mathrm{Psh}(\mathrm{Sm}_{/X}) \xrightarrow{L_X^{\mathrm{mot}}} \mathcal{H}(X).$$

We make the following observation, which will be useful later:

Lemma 2.22. *Let $f\colon X \to Y$ be a morphism between quasi-compact and quasi-separated schemes. Then $f_*\colon \mathrm{Psh}(\mathrm{Sm}_{/X}) \to \mathrm{Psh}(\mathrm{Sm}_{/Y})$ preserves $\mathbb{A}^1$-local equivalences.*

Proof: It suffices to see that, for any $T \in \mathrm{Sm}_{/X}$, the induced morphism

$$f_*(\mathrm{pr}_1)\colon f_*(T \times \mathbb{A}^1_X) \to f_*(T)$$

is an equivalence. Since f_* and $L_Y^{\mathbb{A}^1} = \mathrm{Sing}_Y$ both preserve products, it suffices to see that the canonical morphism

$$\pi\colon f_*(h(\mathbb{A}^1)) \to *$$

to the terminal object is an $\mathbb{A}^1$-local equivalence. For this, we observe that π has a right inverse, induced by applying f_* to the 0-section

$$i\colon X \to \mathbb{A}^1_X$$

in $\mathrm{Psh}(\mathrm{Sm}_{/X})$. We now consider the morphism $H\colon \mathbb{A}^1_Y \times f_*(\mathbb{A}^1_X) \to f_*(\mathbb{A}^1_X)$ which, under adjunction, corresponds to the morphism

$$\begin{aligned} f^*(\mathbb{A}^1_Y \times f_*(\mathbb{A}^1_X)) &\to f^*(\mathbb{A}^1_Y) \times f^* f_*(\mathbb{A}^1_X) \\ &= \mathbb{A}^1_X \times f^* f_*(\mathbb{A}^1_X) \xrightarrow{(\mathrm{id},\varepsilon)} \mathbb{A}^1_X \times \mathbb{A}^1_X \xrightarrow{m} \mathbb{A}^1_X \end{aligned}$$

where ε denotes the counit of the adjunction $f^* \dashv f_*$ and m denotes the multiplication map. Then H defines an $\mathbb{A}^1$-homotopy between $\mathrm{id}_{f_*(\mathbb{A}^1)}$ and the composite $f_*(i) \circ \pi$. Thus the claim follows from Lemma 2.7. □

We will now restrict to the case where $f\colon X \to Y$ is smooth and observe that we will get some extra functoriality:

Construction 2.23. Let $f\colon X \to Y$ be a smooth morphism of quasi-compact and quasi-separated schemes. In this case we get an adjunction

$$F\colon \mathrm{Sm}_{/X} \rightleftarrows \mathrm{Sm}_{/Y} : - \times_X Y,$$

where F is the forgetful functor given by composition with f. It follows that, in the adjunction

$$f^*\colon \mathrm{Psh}(\mathrm{Sm}_{/Y}) \rightleftarrows \mathrm{Psh}(\mathrm{Sm}_{/X}) : f_*,$$

the left adjoint f^* is given by precomposition with F. Hence f^* has a left adjoint $f_\sharp$, given by left Kan-extension. As above, the adjoint $f_\sharp$ preserves colimits and makes the diagram

$$\begin{array}{ccc} \mathrm{Psh}(\mathrm{Sm}_{/X}) & \xrightarrow{f_\sharp} & \mathrm{Psh}(\mathrm{Sm}_{/Y}) \\ \uparrow & & \uparrow \\ \mathrm{Sm}_{/X} & \xrightarrow{F} & \mathrm{Sm}_{/Y} \end{array}$$

commute. We have seen in Example A.19 that $f_\sharp$ preserves Nisnevich-local equivalences. Furthermore, the functor $f_\sharp$ preserves $\mathbb{A}^1$-projections when restricted to $\mathrm{Sm}_{/X}$ and thus it preserves motivic equivalences. It follows that f^* preserves motivic spaces and thus restricts to a functor

$$f^*_{\mathcal{H}}\colon \mathcal{H}(Y) \to \mathcal{H}(X).$$

Hence it has a left adjoint $f^{\mathcal{H}}_\sharp\colon \mathcal{H}(X) \to \mathcal{H}(Y)$ given by the composition

$$\mathcal{H}(Y) \hookrightarrow \mathrm{Psh}(\mathrm{Sm}_{/Y}) \xrightarrow{f_\sharp} \mathrm{Psh}(\mathrm{Sm}_{/X}) \xrightarrow{L^{\mathrm{mot}}_X} \mathcal{H}(X).$$

In particular, we see that $f^*_{\mathcal{H}}$ preserves all small colimits.

Proposition 2.24 (Smooth base change)**.** *Let*

$$\begin{array}{ccc} X \times_S T & \xrightarrow{\bar{g}} & T \\ \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & S \end{array}$$

be a cartesian square of quasi-compact and quasi-separated schemes and assume that f is smooth. Then the base change natural transformations

$$f^*_{\mathcal{H}} g^{\mathcal{H}}_* \to \bar{g}^{\mathcal{H}}_* \bar{f}^*_{\mathcal{H}}$$

and

$$\bar{f}^{\mathcal{H}}_\sharp \bar{g}^*_{\mathcal{H}} \to g^*_{\mathcal{H}} f^{\mathcal{H}}_\sharp$$

are invertible.

Proof: Since the second natural transformation is the mate of the first one, it suffices to see that the second natural transformation is an equivalence. This base change natural transformation is given by applying L^{mot}_X to the analogous natural transformation on the presheaf-level. Since all functors in discussion preserve colimits, in order to prove the claim on the presheaf-level, it suffices to check this for representable presheaves $h(K)$ for some $K \in \mathrm{Sm}_{/T}$. But in this case, the natural transformation is just

$$\begin{array}{rcl} \bar{f}_\sharp \bar{g}^*(h(K)) & = & \\ h((X \times_S T) \times_T K) & \xrightarrow{\cong} & h(K \times_S X) \\ & \cong & g^* f_\sharp(h(K)) \end{array}$$

and we get the claim. □

2.25. Let $f\colon X \to S$ be a smooth morphism of schemes. Let $A, B \in \mathcal{H}(S)$ and $C \in \mathcal{H}(X)$. Assume we are given morphisms $B \to A$ and $C \to f^*_{\mathcal{H}}(A)$. Then we have a natural morphism

$$\phi\colon f^{\mathcal{H}}_\sharp(f^*_{\mathcal{H}} B \times_{f^*_{\mathcal{H}} A} C) \to f^{\mathcal{H}}_\sharp f^*_{\mathcal{H}} B \times_{f^{\mathcal{H}}_\sharp f^*_{\mathcal{H}} A} f^{\mathcal{H}}_\sharp C \to B \times_A f^{\mathcal{H}}_\sharp C,$$

where the second arrow is induced by the counit $\varepsilon\colon f^{\mathcal{H}}_\sharp f^*_{\mathcal{H}} \to \mathrm{id}$.

Proposition 2.26 (Smooth projection formula)**.** *The morphism ϕ from above is an equivalence.*

Proof: Using Remark 2.19, it is easy to see that ϕ is given by applying L^{mot}_S to the analogous natural morphism on the presheaf-level. So it suffices to see the claim for presheaves. We have a canonical identification

$$\mathrm{Psh}(\mathrm{Sm}_{/X}) \simeq \mathrm{Psh}((\mathrm{Sm}_{/S})_{/X}) \simeq \mathrm{Psh}(\mathrm{Sm}_{/S})_{/h(X)},$$

under which the morphism $f_\sharp$ corresponds to the forgetful functor

$$\mathrm{Psh}(\mathrm{Sm}_{/S})_{/h(X)} \to \mathrm{Psh}(\mathrm{Sm}_{/S})$$

and $f^*\colon \mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathrm{Psh}(\mathrm{Sm}_{/S})_{/h(X)}$ corresponds to the assignment

$$\mathcal{F} \mapsto \mathcal{F} \times h(X).$$

Thus the above natural transformation is given by the canonical morphism

$$(B \times h(X)) \times_{A \times h(X)} C \to B \times_A C$$

induced by the projections, which is easily seen to be an equivalence. □

2.4. Closed Immersions

In this section we will study the properties of the push-forward functor $i_*^{\mathcal{H}}$ along a closed immersion $i\colon Z \hookrightarrow X$. We will roughly follow [Kha16, §1.7]. Our goal is the following result:

Proposition 2.27. *Let $i\colon Z \hookrightarrow X$ be a closed immersion. Then $i_*^{\mathcal{H}}$ preserves weakly contractible colimits.*

We will need the following geometric input ([Gro67, Proposition 18.1.1]):

Proposition 2.28. *Let $i\colon Z \hookrightarrow X$ be a closed immersion. Let $f\colon T \to Z$ be an étale morphism and let $x \in T$ be a point. Then there exists an open subscheme $U \subset T$ containing x and an étale X-scheme V such that $V \times_X Z \cong U$ as Z-schemes.*

Construction 2.29. Let C be a small ∞-category with an initial object $\emptyset$. We will denote by $\mathrm{Psh}_\emptyset(C)$ the full subcategory of $\mathrm{Psh}(C)$ spanned by those presheaves $\mathcal{F}$ such that $\mathcal{F}(\emptyset) \simeq *$. One observes that the inclusion

$$\mathrm{Psh}_\emptyset(C) \hookrightarrow \mathrm{Psh}(C)$$

admits a left adjoint, that takes a presheaf $\mathcal{F}$ to the presheaf $\mathcal{F}_\emptyset$, which is given as follows: If an object U in C is initial then $\mathcal{F}_\emptyset(U) \simeq *$ and else $\mathcal{F}_\emptyset(U) \simeq \mathcal{F}(U)$. We will denote this left adjoint by $L_\emptyset$. Furthermore, observe that we have a localization functor

$$L^{\mathrm{mot}}_{S,\emptyset}\colon \mathrm{Psh}_\emptyset(\mathrm{Sm}_{/Z}) \to \mathcal{H}(Z),$$

which is a left adjoint to the inclusion $\mathcal{H}(Z) \hookrightarrow \mathrm{Psh}_\emptyset(\mathrm{Sm}_{/Z})$. Note that $L^{\mathrm{mot}}_{S,\emptyset}$ can be described as the composition of the inclusion $\mathrm{Psh}_\emptyset(\mathrm{Sm}_{/Z}) \to \mathrm{Psh}(\mathrm{Sm}_{/Z})$ with L^{mot}_S.

Lemma 2.30. *Let $i\colon Z \hookrightarrow X$ be a closed immersion. Then the composite*

$$\Psi\colon \mathrm{Psh}_\emptyset(\mathrm{Sm}_{/Z}) \hookrightarrow \mathrm{Psh}(\mathrm{Sm}_{/Z}) \xrightarrow{i_*} \mathrm{Psh}(\mathrm{Sm}_{/X})$$

preserves motivic equivalences.

Proof: By Lemma 2.22, it suffices to see that Ψ takes $L_\emptyset(R \hookrightarrow h(Q))$ to a motivic equivalence, for any $Q \in \mathrm{Sm}_{/Z}$ and any Nisnevich covering sieve $R \hookrightarrow h(Q)$. We will show this by proving that

$$\Psi(L_\emptyset(R \hookrightarrow h(Q)))$$

is a Nisnevich-local equivalence. Since representables generate $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/X})$ under colimits and since colimits are universal, it suffices to see that, for any $h(T) \to \Psi(h(Q))$, the pullback

$$S := \Psi(L_\emptyset(R)) \times_{\Psi(h(Q))} h(T) \to h(T)$$

is a Nisnevich covering sieve. Since R is a Nisnevich covering sieve, there is a finite collection $\{f_j \colon U_j \to Q\}_{j \in J}$ of étale morphisms in R such that the induced morphism

$$\coprod_{j \in J} U_j \to Q$$

is a Nisnevich covering morphism. Note that, by the Yoneda lemma, a morphism $h(T) \to \Psi(h(Q))$ as above is given by a morphism

$$\alpha \colon T \times_X Z \to Q$$

over Z. Unwinding the definitions, we see that, for $K \in \mathrm{Sm}_{/X}$, the set $S(K)$ consists of all morphisms $f \colon K \to T$ in $\mathrm{Sm}_{/X}$ such that after pulling back along i and composing with α, the induced morphism

$$K \times_X Z \xrightarrow{f \times_X Z} T \times_X Z \xrightarrow{\alpha} Q$$

is in R, or $K \times_X Z = \emptyset$. Let us write $g_j \colon U_j \times_Q (T \times_X Z) \to T \times_X Z$ for the pullback of f_j along α. By refining with open subschemes if necessary, Proposition 2.28 allows us to assume that for every $j \in J$ there is an étale T-scheme $h_j \colon V_j \to T$ such that the pullback along $i \times_X T$ is isomorphic to $U_j \times_Q (T \times_X Z)$ over $T \times_X Z$. We now consider the sieve $\bar{S}$ on T generated by $\{g_j \colon V_j \to T\}$ and $T \setminus (T \times_X Z) \hookrightarrow T$. This is a Nisnevich covering sieve, as the canonical morphism

$$\left(\coprod_{j \in J} V_j\right) \amalg T \setminus (T \times_X Z) \to T$$

is clearly a Nisnevich covering morphism. Furthermore, we get $\bar{S} \subseteq S$ by construction, so S is a Nisnevich covering sieve as well, and we get the claim. □

Proof of Proposition 2.27: We will write $j \colon \mathcal{H}(Z) \to \mathrm{Psh}(Z)$ for the inclusion. Let $D \colon I \to \mathcal{H}(Z)$ be a weakly contractible diagram. Let C be the colimit of this diagram in $\mathrm{Psh}(\mathrm{Sm}_{/Z})$ and observe that, since I is weakly contractible, we have $C \in \mathrm{Psh}_\emptyset(\mathrm{Sm}_{/Z})$. We have a canonical map

$$\eta \colon C \to j(\operatorname*{colim}_i D(i))$$

in $\mathrm{Psh}(\mathrm{Sm}_{/Z})$, which clearly is a motivic equivalence. By Lemma 2.30, it follows that the canonical map

$$\operatorname*{colim}_i \Psi(j(D(i))) \simeq \Psi(C) \to \Psi(j(\operatorname*{colim}_i D(i)))$$

in $\mathrm{Psh}(\mathrm{Sm}_{/X})$ is a motivic equivalence (here we use the notation from Lemma 2.30). Thus, by applying L_X^{mot}, we get that the canonical morphism

$$\operatorname*{colim}_i i_*^{\mathcal{H}}(D(i)) \simeq L_X^{\mathrm{mot}}(\operatorname*{colim}_i \Psi(j(D(i)))) \to i_*^{\mathcal{H}}(\operatorname*{colim}_i D(i))$$

is an equivalence, as desired. □

2.5. The Localization Theorem

For this section, we fix a noetherian scheme of finite Krull dimension S, a closed immersion $i\colon Z \to S$ and the complementary open immersion $j\colon U \to S$. For $X \in \mathrm{Sm}_{/S}$, we will write $X_Z := Z \times_S X$ and $X_U := U \times_S X$. We start with the following trivial observation:

Lemma 2.31. *The composite $i_{\mathcal{H}}^* j_\sharp^{\mathcal{H}} \colon \mathcal{H}(U) \to \mathcal{H}(Z)$ is equivalent to the constant functor on an initial object of $\mathcal{H}(Z)$.*

Proof: It suffices to see that, for any $T \in \mathrm{Sm}_{/U}$, the motivic space $i_{\mathcal{H}}^* j_\sharp^{\mathcal{H}}(L_U^{\mathrm{mot}} T)$ is equivalent to the initial object. Since

$$i_{\mathcal{H}}^* j_\sharp^{\mathcal{H}}(L_U^{\mathrm{mot}} T) \simeq L_Z^{\mathrm{mot}}(i^* j_\sharp U),$$

the claim follows as $i^* j_\sharp U \simeq Z \times_S T = \emptyset$. □

2.32. Consider the following diagram

$$\begin{array}{ccc} j_\sharp^{\mathcal{H}} j_{\mathcal{H}}^* \mathcal{F} & \xrightarrow{\ \varepsilon\ } & \mathcal{F} \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle \eta} \\ U & \xrightarrow[\ b\ ]{} & i_*^{\mathcal{H}} i_{\mathcal{H}}^* \mathcal{F} \end{array}$$

in $\mathcal{H}(S)$, where ε and η are the counit and unit, respectively, of the respective adjunctions. The map a corresponds, under adjunction, to the unique map $j_{\mathcal{H}}^* \mathcal{F} \to U$ in $\mathcal{H}(U)$ and b corresponds, under adjunction, to the canonical map $\emptyset = i_{\mathcal{H}}^*(U) \to i_{\mathcal{H}}^*(\mathcal{F})$. Furthermore, the above lemma shows that the square indeed commutes, since there is (up to equivalence) only one morphism $j_\sharp^{\mathcal{H}} j_{\mathcal{H}}^* \mathcal{F} \to i_*^{\mathcal{H}} i_{\mathcal{H}}^* \mathcal{F}$.

The goal of this section is to prove the following theorem, for which we will follow the argument given in [MV99].

Theorem 2.33. *The commutative square*

$$\begin{array}{ccc} j_\sharp^{\mathcal{H}} j_{\mathcal{H}}^* \mathcal{F} & \xrightarrow{\varepsilon} & \mathcal{F} \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle \eta} \\ U & \xrightarrow[b]{} & i_*^{\mathcal{H}} i_{\mathcal{H}}^* \mathcal{F} \end{array}$$

is a pushout square in $\mathcal{H}(S)$*.*

Remark 2.34. In the proof of Theorem 2.33, it will become apparent why our definition of the unstable motivic homotopy category is the right one for our purposes. The argument will heavily depend on the fact that we are defining motivic spaces as certain presheaves on the category $\mathrm{Sm}_{/S}$ of *smooth* schemes finitely presented over our given base S, as opposed to for example *all* schemes finitely presented over S. It will also be important that the descent conditions that we impose are at least as strong as Nisnevich descent.

Our next goal is to show that motivic equivalences can be, just like Nisnevich-local equivalences, detected stalkwise.

2.35. Let U be a noetherian scheme and let $u \in U$ be a point. Write U_u^h for the henselization at u and $p_u\colon U_u^h \to U$ for the canonical morphism. For a smooth scheme T over U_u^h, it follows from [Sta20, Tag 01ZM] and [Sta20, Tag 0C0C] that there is some Nisnevich neighbourhood $(U_0, u_0) \to (U, u)$ and a smooth morphism $p\colon T_0 \to U_0$ such that the diagram

$$\begin{array}{ccc} T & \longrightarrow & T_0 \\ \downarrow & & \downarrow{\scriptstyle p} \\ U_u^h & \longrightarrow & U_0 \end{array}$$

is a pullback square. For $(V, v) \in \mathrm{Nbhd}(U_0, u_0)$, we define

$$T_0^{(V,v)} := T_0 \times_{U_0} V$$

and then we get

$$T \cong \lim_{(V,v)\in \mathrm{Nbhd}(U_0,u_0)} T_0^{(V,v)}.$$

Write $\mathrm{ev}_T\colon \mathrm{Psh}(\mathrm{Sm}_{/U_u^h}) \to \mathcal{S}$ for the evaluation functor at T and consider the composite

$$\mathrm{Psh}(\mathrm{Sm}_{/S}) \xrightarrow{p_u^*} \mathrm{Psh}(\mathrm{Sm}_{/U_u^h}) \xrightarrow{\mathrm{ev}_T} \mathcal{S}.$$

We claim that the latter is equivalent to the composite

$$\mathrm{Psh}(\mathrm{Sm}_{/S}) \to \mathrm{Fun}(\mathrm{Nbhd}(U_0, u_0)^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\mathrm{colim}} \mathcal{S},$$

where the first functor is given by restricting along

$$\begin{aligned} \mathrm{Nbhd}(U_0, u_0)^{\mathrm{op}} &\to \mathrm{Sm}_{/S}^{\mathrm{op}} \\ (V, v) &\mapsto T_0^{(V,v)}. \end{aligned}$$

This follows since we may restrict along Yoneda embedding, where the assertion is clear.

We obtain the following consequence:

Lemma 2.36. *In the situation above, the functor* $p_u^*\colon \mathrm{Psh}(\mathrm{Sm}_{/U}) \to \mathrm{Psh}(\mathrm{Sm}_{/U_u^h})$ *preserves Nisnevich-local and* $\mathbb{A}^1$*-invariant presheaves.*

Proof: We start by showing that p_u^* preserves $\mathbb{A}^1$-invariance. As above, we observe that, for $T \in \mathrm{Sm}_{/U_u^h}$, we find a Nisnevich neighbourhood $(U_0, u_0) \to (U, u)$ and a smooth U_0-scheme T_0 such that

$$T \cong \lim_{(V,v)\in \mathrm{Nbhd}(U_0,u_0)} T_0^{(V,v)}.$$

So the projection $T \times \mathbb{A}^1 \to T$ is identified with the limit of the morphisms

$$T_0^{(V,v)} \times \mathbb{A}^1 \to T_0^{(V,v)}$$

and thus, for an $\mathbb{A}^1$-invariant presheaf $\mathcal{F} \in \mathrm{Psh}^{\mathbb{A}^1}(\mathrm{Sm}_{/U})$, the induced morphism

$$p_u^*\mathcal{F}(T) \to p_u^*\mathcal{F}(T \times \mathbb{A}^1)$$

can by 2.35 be identified with the colimit of the induced morphisms

$$\mathcal{F}(T_0^{(V,v)}) \to \mathcal{F}(T_0^{(V,v)} \times \mathbb{A}^1)$$

and therefore is an $\mathbb{A}^1$-local equivalence. To show that p_u^* preserves Nisnevich-local objects, we observe that it follows from [Sta20, Tag 01ZM] and the results in [Sta20, Tag 081C], that any Nisnevich square

$$\begin{array}{ccc} W \times_T Q & \longrightarrow & Q \\ \downarrow & & \downarrow \\ W & \longrightarrow & T \end{array}$$

in $\mathrm{Sm}_{/U_u^h}$ can be written as a limit of Nisnevich squares of smooth schemes over suitable Nisnevich neighbourhoods of u. Then the claim follows from 2.35 since filtered colimits commute with finite limits. □

Proposition 2.37. *Let* $\alpha\colon \mathcal{F} \to \mathcal{G}$ *be a morphism in* $\mathrm{Psh}(\mathrm{Sm}_{/S})$ *such that, for every* $(U, u) \in \mathrm{Sm}_{/S\,*}$*, the morphism* $(f \circ p_u)^*(\alpha)$ *in* $\mathrm{Psh}(U_u^h)$ *is a motivic equivalence, where* $f\colon U \to S$ *denotes the given morphism. Then* α *is a motivic equivalence.*

Proof: By assumption, we have that

$$L^{\mathrm{mot}}_{U_u^h}((f \circ p_u)^*(\alpha)) \simeq (f \circ p_u)^*_{\mathcal{H}}(L^{\mathrm{mot}}_S(\alpha))$$

is an equivalence. But since f^* and p_u^* both preserve $\mathbb{A}^1$-invariant and Nisnevich-local presheaves by Construction 2.23 and Lemma 2.36, the functor $(f \circ p_u)^*_{\mathcal{H}}$ is given by simply restricting $(f \circ p_u)^*$ to $\mathcal{H}(S)$. Therefore the claim follows from Proposition 1.40 and Theorem 1.46. □

Definition 2.38. Let $X \in \mathrm{Sm}_{/S}$ such that there is a section $t\colon Z \to X_Z$ of the projection $X_Z := Z \times_S X \to Z$. We then have an induced map $S \to i_* X_Z$ that, under adjunction, corresponds to t. So we can define a presheaf of sets on $\mathrm{Sm}_{/S}$ via the formula

$$\Phi_S(X, t) := (X \amalg_{X_U} U) \times_{i_* X_Z} S.$$

Unwinding the definitions, we get the following formula for $Y \in \mathrm{Sm}_{/S}$:

$$\Phi_S(X,t)(Y) = \begin{cases} \mathrm{map}_S(Y, X) \times_{\mathrm{map}_Z(Y_Z, X_Z)} * & \text{if } Y_Z \neq \emptyset \\ * & \text{else} \end{cases}$$

Here the map $* \to \mathrm{map}_Z(Y_Z, X_Z)$ selects the composition $Y_Z \to Z \xrightarrow{t} X_Z$. Note that this construction is functorial in the following sense: If $f\colon X' \to X$ is a morphism in $\mathrm{Sm}_{/S}$ such that there are sections $t\colon Z \to X_Z$ and $t'\colon Z \to X'_Z$ with $f_Z \circ t' = t$, then there is a canonical morphism

$$\varphi_f\colon \Phi_S(X', t') \to \Phi_S(X, t),$$

which is induced by composition with f.

Remark 2.39. Note that the explicit formula above makes sense for any S-scheme Y and not only for smooth S-schemes. So we can canonically define a presheaf

$$\Phi_S(X, t)\colon \mathrm{Sch}_{/Y} \to \mathrm{Set}$$

that agrees with the one above when restricted to $\mathrm{Sm}_{/S}$.

Let us recall the following fact about henselian local rings:

Lemma 2.40. *Let $(R, \mathfrak{m})$ be a henselian local ring. Suppose we are given a solid commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}\,(R/\mathfrak{m}) & \longrightarrow & X' \\ \downarrow & \nearrow & \downarrow p \\ \mathrm{Spec}\,(R) & \longrightarrow & X \end{array}$$

of schemes, where p is étale. Then there is a unique dotted arrow making the diagram commute.

Proof: See [Sta20, Tag 08HQ] for the affine version one immediately reduces to as $\mathrm{Spec}\,(R)$ is local. □

Lemma 2.41. *Let $f\colon X' \to X$ be an étale morphism in $\mathrm{Sm}_{/S}$ such that there are sections $t\colon Z \to X_Z$ and $t'\colon Z \to X'_Z$ with $f_Z \circ t' = t$. Then the induced morphism*

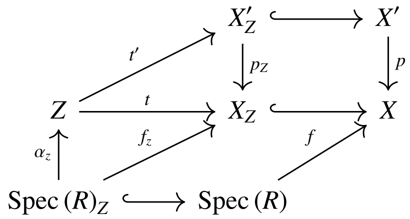
$$\varphi_f\colon \Phi_S(X', t') \to \Phi_S(X, t)$$

is a Nisnevich-local equivalence.

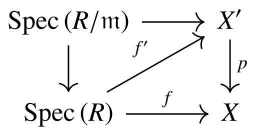
Proof: By Theorem 1.46, it suffices to see that φ_f is a stalkwise equivalence. So let us pick a henselian local ring $(R, \mathfrak{m})$, equipped with a morphism $\alpha\colon \operatorname{Spec}(R) \to S$. We want to show that the canonical morphism

$$\Phi_S(X', t')(\operatorname{Spec}(R)) \to \Phi_S(X, t)(\operatorname{Spec}(R))$$

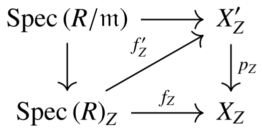
is an isomorphism. If $Z = \emptyset$, the claim is obvious, so we assume that $Z \neq \emptyset$. An element in the right hand side is given by a morphism $f\colon \operatorname{Spec}(R) \to X$ such that its restriction f_Z is given by $t \circ \alpha_Z$. By the assumption, this gives the commutative diagram



By precomposing with the canonical morphism $\operatorname{Spec}(R/\mathfrak{m}) \to \operatorname{Spec}(R)_Z$ in the bottom left corner, Lemma 2.40 yields a morphism $f'\colon \operatorname{Spec}(R) \to X'$ which fits into the commutative diagram



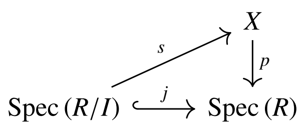
Pulling back to Z yields the commutative diagram



and it follows that $f'_Z = t' \circ \alpha_Z$ by the uniqueness in Lemma 2.40. Thus f' defines an element in $\Phi_S(X, t')(\operatorname{Spec}(R))$ which is a preimage of f. Since any other preimage of f agrees with f' when pulled back to Z, it follows that f' is the only preimage, again by the uniqueness in Lemma 2.40. Thus we get the claim. □

We will also need the following variation of Lemma 2.40 for smooth morphisms:

Lemma 2.42. *Let X be a smooth scheme over a henselian local ring $(R, \mathfrak{m})$. Let $I \subseteq R$ be an ideal and assume we are given a commutative triangle*



Then there exists a section $s'\colon \operatorname{Spec}(R) \to X$ of p such that $s' \circ j = s$.

Proof: Let us by $x \in \operatorname{Spec}(R/I) \subseteq \operatorname{Spec}(R)$ denote the closed point corresponding to the maximal ideal. By [Sta20, Tag 054L], there are $\operatorname{Spec}(A) \subseteq X$ and $\operatorname{Spec}(B) \subseteq \operatorname{Spec}(R)$ affine open with $s(x) \in \operatorname{Spec}(A)$ and $p(\operatorname{Spec}(A)) \subseteq \operatorname{Spec}(B)$ and a natural number d such that we have a factorization

$$\begin{tikzcd} X \arrow[d, "p"] & \operatorname{Spec}(A) \arrow[l] \arrow[r, "f"] \arrow[d, "p'"] & \mathbb{A}^d_B \arrow[dl, "\mathrm{pr}"] \\ \operatorname{Spec}(R) & \operatorname{Spec}(B) \arrow[l] & \end{tikzcd}$$

with pr the canonical projection and f étale. Since $\operatorname{Spec}(R)$ is local and $x \in \operatorname{Spec}(B)$, it follows that $\operatorname{Spec}(B) = \operatorname{Spec}(R)$. As $\operatorname{Spec}(R/I)$ is local as well, it follows that s factors through $\operatorname{Spec}(A)$ and thus we have reduced the situation to the case where we have the diagram

$$\begin{tikzcd} & \operatorname{Spec}(A) \arrow[r, "f"] \arrow[d, "p'"] & \mathbb{A}^d_R \arrow[dl, "\mathrm{pr}"] \\ \operatorname{Spec}(R/I) \arrow[ur, "s"] \arrow[r, hook] & \operatorname{Spec}(R) & \end{tikzcd}$$

By the universal property of the polynomial algebra, we can extend the morphism $f \circ s$ to a section $s_0 \colon \operatorname{Spec}(R) \to \mathbb{A}^d_R$ of pr and get a commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(R/\mathfrak{m}) \arrow[d] \arrow[dr] & \\ \operatorname{Spec}(R/I) \arrow[r, "s"] \arrow[d] & \operatorname{Spec}(A) \arrow[d, "f"] \\ \operatorname{Spec}(R) \arrow[r, "s_0"] & \mathbb{A}^d_R \end{tikzcd}$$

By Lemma 2.40, we get a morphism $s' \colon \operatorname{Spec}(R) \to \operatorname{Spec}(A)$ making the diagram

$$\begin{tikzcd} \operatorname{Spec}(R/\mathfrak{m}) \arrow[r] \arrow[d] & \operatorname{Spec}(A) \arrow[d, "f"] \\ \operatorname{Spec}(R) \arrow[ur, "s'"] \arrow[r, "s_0"] & \mathbb{A}^d_R \end{tikzcd}$$

commute and by the uniqueness in Lemma 2.40, the diagram

$$\begin{tikzcd} \operatorname{Spec}(R/I) \arrow[r, "s"] \arrow[d] & \operatorname{Spec}(A) \arrow[d, "f"] \\ \operatorname{Spec}(R) \arrow[ur, "s'"] \arrow[r, "s_0"] & \mathbb{A}^d_R \end{tikzcd}$$

commutes as well. Thus s' is the desired section. □

The main ingredient of the proof of Theorem 2.33 is the following:

Proposition 2.43. *For $X \in \mathrm{Sm}_{/S}$ and a section $t\colon Z \to X_Z$ of the canonical morphism $X_Z \to Z$, the presheaf $\Phi_S(X, t)$ is motivically contractible.*

Proof: By Proposition 2.37, we may assume that $S = \mathrm{Spec}(R)$ for a henselian local ring R. Since S is henselian, the section $Z \to X_Z$ extends to a section $s\colon S \to X$ by Lemma 2.42. Using the same arguments as in the proof of Lemma 2.42 and by applying Lemma 2.41, we may assume that X is affine and that there is a commutative diagram

$$\begin{array}{ccccc} & & X & \xrightarrow{f} & \mathbb{A}^d_S \\ & \overset{s}{\nearrow} & \downarrow & \swarrow & \\ Z \longrightarrow & S & = & S & \end{array}$$

where f is étale. The section $f \circ s$ is, on the level of rings, given by a morphism

$$\alpha\colon R[X_1, \dots, X_d] \to R$$

and we can compose with the isomorphism of R-algebras

$$\begin{aligned} R[T_1, \dots, T_d] &\to R[X_1, \dots, X_d] \\ T_i &\mapsto X_i - \alpha(X_i) \end{aligned}$$

to assume that $f \circ s$ is given by the zero section $i_0\colon S \to \mathbb{A}^d_S$. Since f is étale, we can apply Lemma 2.41 again and only have to show that $\Phi_S(\mathbb{A}^d_S, t_0)$ is motivically contractible, where $t_0\colon Z \to \mathbb{A}^d_Z$ is the zero section. For this, observe that, for any $Y \in \mathrm{Sm}_{/S}$, we have that $\Phi_S(\mathbb{A}^d_S, t_0)(Y)$ is a subset of $h(\mathbb{A}^d_S)(Y)$ and the 0-section $i_0\colon S \to \mathbb{A}^d_S$ restricts to a section $s\colon S \to \Phi_S(\mathbb{A}^d_S, t_0)$ of the unique morphism $\Phi_S(\mathbb{A}^d_S, t_0) \to S$. Furthermore, the homotopy

$$\begin{aligned} H\colon \mathbb{A}^1_S \times \mathbb{A}^d_S &\to \mathbb{A}^d_S \\ (\lambda, x) &\mapsto \lambda x \end{aligned}$$

between the identity and the composite $\mathbb{A}^d_S \xrightarrow{\mathrm{pr}} S \xrightarrow{i_0} \mathbb{A}^d_S$ restricts to a homotopy

$$K\colon \mathbb{A}^1_S \times \Phi_S(\mathbb{A}^d_S, t_0) \to \Phi_S(\mathbb{A}^d_S, t_0)$$

between the identity and the composite $\Phi_S(\mathbb{A}^d_S, t_0) \to S \xrightarrow{s} \Phi_S(\mathbb{A}^d_S, t_0)$. This completes the proof. □

Proof of Theorem 2.33: We start by noting that $j^*_{\mathcal{H}}$, $j^{\mathcal{H}}_\sharp$ and $i^*_{\mathcal{H}}$ preserve all colimits as they are left adjoints. Furthermore, the functor $i^{\mathcal{H}}_*$ preserves weakly contractible colimits by Proposition 2.27. Since sifted simplicial sets are weakly contractible ([Lur09, Proposition 5.5.8.7]), it follows from Lemma 2.20 that we may assume that $\mathcal{F}$ is given by $L^{\mathrm{mot}}_S X$ for some smooth S-scheme X. Note that a consequence of Lemma 2.30 is that the diagram

$$\begin{array}{ccc} \mathrm{Psh}_\emptyset(\mathrm{Sm}_{/Z}) & \xrightarrow{i_*} & \mathrm{Psh}_\emptyset(\mathrm{Sm}_{/S}) \\ \downarrow{\scriptstyle L^{\mathrm{mot}}_Z} & & \downarrow{\scriptstyle L^{\mathrm{mot}}_S} \\ \mathcal{H}(Z) & \xrightarrow{i^{\mathcal{H}}_*} & \mathcal{H}(S) \end{array}$$

commutes. So we compute:

$$i_*^{\mathcal{H}} i_{\mathcal{H}}^* L^{\mathrm{mot}}(X) \simeq i_*^{\mathcal{H}}(L_Z^{\mathrm{mot}}(X_Z)) \simeq L_S^{\mathrm{mot}}(i_* X_Z).$$

Furthermore, we have that $j_\sharp^{\mathcal{H}} j_{\mathcal{H}}^* L_S^{\mathrm{mot}} X \simeq L_S^{\mathrm{mot}}(X_U)$ and so it suffices to see that the map of presheaves

$$X \amalg_{X_U} U \to i_* X_Z$$

is a motivic equivalence. By Remark 2.19, it suffices to see that, for any $p\colon Y \to S \in \mathrm{Sm}_{/S}$ equipped with a map $Y \to i_* X_Z$, the pulled back map

$$\psi\colon (X \amalg_{X_U} U) \times_{i_* X_Z} Y \to Y$$

is a motivic equivalence. Consider the following cartesian square:

$$\begin{array}{ccc} Y_Z & \xrightarrow{i'} & Y \\ \big\downarrow{\scriptstyle p'} & & \big\downarrow{\scriptstyle p} \\ Z & \xrightarrow{i} & S \end{array}$$

Using the smooth projection formula and smooth base change on the presheaf-level (see the proofs of Proposition 2.26 and Proposition 2.24), we see that ψ is equivalent to applying $p_\sharp$ to the projection

$$(p^* X \amalg_{p^* X_U} p^* U) \times_{i'_* i'^* p^* X} Y \to Y.$$

Indeed, by smooth base change, we get that

$$i'_* i'^* p^* X \simeq i'_* p'^* i^* X \simeq p^* i_* i^* X$$

and thus, by the smooth projection formula, we have

$$\begin{aligned} p_\sharp((p^* X \amalg_{p^* X_U} p^* U) \times_{i'_* i'^* p^* X} Y) &\simeq p_\sharp(p^*(X \amalg_{X_U} U) \times_{p^* i_* i^* X} Y) \\ &\simeq (X \amalg_{X_U} U) \times_{i_* i^* X} Y. \end{aligned}$$

Since $p_\sharp$ preserves motivic equivalence, this shows that we may assume that $Y = S$. In this case, the morphism $S \to i_* X_Z$ corresponds under adjunction to a section $t\colon Z \to X_Z$ of the canonical morphism $X_Z \to Z$. So it suffices to show that the canonical map

$$\Phi_S(X, t) \to S$$

is a motivic equivalence, i.e. that $\Phi_S(X, t)$ is motivically contractible. But this is precisely Proposition 2.43 and the claim follows. □

Remark 2.44. Note that the above theorem does not hold it we do not impose $\mathbb{A}^1$-invariance. By this we mean that the analogous diagram

$$\begin{array}{ccc} j_\sharp j^* \mathcal{F} & \longrightarrow & \mathcal{F} \\ \big\downarrow & & \big\downarrow \\ U & \longrightarrow & i^* i_* \mathcal{F} \end{array} \tag{1}$$

of sheaves is not cocartesian in the ∞-topos $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$. We can for example pick a noetherian henselian local ring $(R, \mathfrak{m})$ of finite Krull dimension. We consider the closed subscheme Z given by the quotient $R \to R/\mathfrak{m}$. Let U be the complement and let j and i be the associated open and closed immersion, respectively. We now consider the sheaf represented by the affine line $\mathbb{A}^1_R$. Then the square

$$\begin{array}{ccc} \mathbb{A}^1_U & \longrightarrow & \mathbb{A}^1_R \\ \downarrow & & \downarrow \\ U & \longrightarrow & i^*(\mathbb{A}^1_Z) \end{array}$$

is not cocartesian in $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Spec}\,(R))$. To see this, we can take the stalk at the closed point of $\mathrm{Spec}\,(R)$, which is equivalent to applying the global sections functor, as $\mathrm{Spec}\,(R)$ is henselian local. Then the induced square is

$$\begin{array}{ccc} \emptyset & \longrightarrow & R \\ \downarrow & & \downarrow \\ \emptyset & \longrightarrow & R/\mathfrak{m} \end{array}$$

which is clearly not a pushout square. The proof of Theorem 2.33 shows how imposing $\mathbb{A}^1$-invariance fixes this behaviour.

However, an easy adaption of the arguments in this section shows that the analogue of Theorem 2.33 holds if we restrict ourselves to the small Nisnevich site and work with the ∞-topos S_{Nis} instead.

3. Algebraic K-Theory of Stable ∞-Categories

In this chapter we will study the algebraic K-theory of small stable ∞-categories. In section 3.1 we will introduce the $S_\bullet$-construction and use it to define the algebraic K-theory of small pointed ∞-categories with finite colimits. We will then focus on the special case of stable ∞-categories. One of the main features of the algebraic K-theory functor is that it sends certain exact sequences of small stable ∞-categories to fiber sequences of K-theory spaces. To make such a statement precise, we have to introduce and study suitable notions of exact sequences at first, which will be the goal of section 3.2. In section 3.3 we will then deduce the above mentioned statement (see Theorem 3.43). In section 3.4 we will construct the non-connective K-theory spectrum and use the results of section 3.3 to show that it sends exact sequences of stable ∞-categories to fiber sequences of spectra.

3.1. The $S_\bullet$-Construction

In this section we will roughly follow [Lur14, Lecture 16].

Definition 3.1. Let C be a pointed ∞-category with finite colimits and let P be a poset. We define

$$P^{(2)} := \{(i, j) \in P \times P \mid i \leq j\}$$

Then a *P-gapped object* of C is a functor $F\colon P^{(2)} \to C$ satisfying:

i) For all $i \in P$ we have $F(i,i) \simeq *$, where $*$ is a terminal object of C.

ii) For all $i \leq j \leq k$ in P, the square

$$\begin{array}{ccc} F(i,j) & \longrightarrow & F(i,k) \\ \downarrow & & \downarrow \\ F(j,j) & \longrightarrow & F(j,k) \end{array}$$

is a pushout square. In other words,

$$F(i,j) \to F(i,k) \to F(j,k)$$

is a cofiber sequence.

We will denote the full subcategory of $\mathrm{Fun}(P^{(2)}, C)$ spanned by the P-gapped objects by $\mathrm{Gap}_P(C)$. Note that a morphism of posets $P \to Q$ induces a morphism $\mathrm{Gap}_Q(C) \to \mathrm{Gap}_P(C)$ of P-gapped objects.

We will denote by $S_n(C)$ the maximal subgroupoid of $\mathrm{Gap}_{[n]}(C)$. We thus obtain simplicial Kan-complex $S_\bullet(C)$. We call $S_\bullet(C)$ the *Waldhausen construction* of C.

Remark 3.2. Spelling out the above definition, we see that the ∞-category $\mathrm{Gap}_{[n]}(C)$ is the ∞-category of diagrams of the form

$$\begin{array}{ccccccc} X_{0,0} & \longrightarrow & X_{0,1} & \longrightarrow & \dots & \longrightarrow & X_{0,n} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & X_{1,1} & \longrightarrow & \dots & \longrightarrow & X_{1,n} \\ & & & & \downarrow & & \downarrow \\ & & & & \dots & \longrightarrow & \dots \\ & & & & & & \downarrow \\ & & & & & & X_{n,n} \end{array}$$

where every square is a pushout square and all objects on the diagonal are equivalent to the zero object.

Definition 3.3. Let C be a pointed ∞-category with finite colimits. Consider the geometric realization $|S_\bullet(C)|$. It has a canonical (up to contractible choice) base point t, induced by the canonical map $* \simeq S_0(C) \to |S_\bullet(C)|$. We define $K(C)$ to be the loop space $\Omega(|S_\bullet(C)|)$. We call $K(C)$ the *K-theory space* of C and define the *n-th K-group of C* to be

$$K_n(C) := \pi_n(K(C), t) \simeq \pi_{n+1}(|S_\bullet(C)|, t).$$

Remark 3.4.

i) Recall that, for a simplicial Kan-complex $U_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{sSet}$, the geometric realization $|U_\bullet|$ agrees with the homotopy type represented by the diagonal simplicial set

$$\begin{aligned} D(U_\bullet)\colon \mathbf{\Delta}^{\mathrm{op}} &\to \mathrm{Set} \\ [n] &\mapsto (U_n)_n. \end{aligned}$$

However, note that this simplicial set in general is not a Kan-complex.

ii) In the situation above, it is not hard to see that the diagonal simplicial set $D(S_\bullet(C))$ associated to $S_\bullet(C)$ does in fact have the horn lifting property with respect to all horns Λ^n_j, with $n \leq 2$. It follows that the fundamental group $\pi_1(|S_\bullet(C)|, *) = K_0(C)$ can be identified with the set of equivalence classes of 1-simplices $\alpha\colon \Delta^1 \to D(S_\bullet(C))$. The group structure is given as follows: We have $[\alpha] \cdot [\beta] = [\gamma]$ if there is a 2-simplex

$$\sigma\colon \Delta^2 \to D(S_\bullet(C))$$

such that

$$\begin{aligned} \sigma|_{\Delta^{\{0,1\}}} &= \alpha, \\ \sigma|_{\Delta^{\{1,2\}}} &= \beta \\ \text{and } \sigma|_{\Delta^{\{0,2\}}} &= \gamma. \end{aligned}$$

Unwinding the definitions, we see that equivalence classes of 1-simplices can be identified with equivalence classes of objects in C and $[X] \cdot [Y] = [Z]$ if and only if there is a cofiber sequence

$$X \to Z \to Y$$

in C. The two canonical cofiber sequences

$$X \to X \amalg Y \to Y$$

and

$$Y \to X \amalg Y \to X$$

furthermore show that $K_0(C)$ is abelian. Thus $K_0(C)$ is isomorphic to the free abelian group generated by the set of equivalence classes of objects in C modulo the relation that $[X] + [Y] = [Z]$, whenever there is a cofiber sequence

$$X \to Z \to Y.$$

Remark 3.5. Let C and $\mathcal{D}$ be pointed ∞-categories with finite colimits and let $F\colon C \to \mathcal{D}$ be a functor which preserves finite colimits. Then, for any $n \in \mathbb{N}$, the induced functor

$$\mathrm{Fun}([n]^{(2)}, C) \to \mathrm{Fun}([n]^{(2)}, \mathcal{D})$$

restricts to a functor $\mathrm{Gap}_{[n]}(C) \to \mathrm{Gap}_{[n]}(\mathcal{D})$. Thus we get an induced map $|S_n(C)| \to |S_n(\mathcal{D})|$ and finally a map $K(C) \to K(\mathcal{D})$ which we will denote by $K(F)$.

Construction 3.6. Observe that the coproduct functor $\amalg\colon C \times C \to C$ is right exact and thus induces a functor

$$+\colon |S_\bullet(C)| \times |S_\bullet(C)| \cong |S_\bullet(C \times C)| \to |S_\bullet(C)|.$$

Note that, if we apply π_1 to the above map, we recover the addition map of $K_0(C)$. Furthermore, if we are given two right exact functors $F, G\colon C \to \mathcal{D}$, we get an induced functor

$$|S_\bullet(F)| + |S_\bullet(G)|\colon |S_\bullet(C)| \to |S_\bullet(\mathcal{D})|$$

given be applying $|S_\bullet(-)|$ to the composite

$$C \xrightarrow{(F,G)} \mathcal{D} \times \mathcal{D} \xrightarrow{\amalg} \mathcal{D}.$$

We will now quickly note the following useful observation (see [BGT13, Lemma 7.3]):

Lemma 3.7. *Restriction along the inclusion*

$$\begin{aligned} j\colon [n-1] &\hookrightarrow [n]^{(2)} \\ i &\mapsto (0, i+1) \end{aligned}$$

induces an equivalence

$$\mathrm{Gap}_{[n]}(C) \xrightarrow{\simeq} \mathrm{Fun}(\Delta^{n-1}, C).$$

In particular, the ∞-category $\mathrm{Gap}_{[n]}(C)$ *is again stable.*

Notation 3.8. We will write $\mathrm{Cat}_\infty^{\mathrm{ex}}$ for the subcategory of Cat_∞ that is spanned by the small stable ∞-categories and exact functors between them.

Construction 3.9. Let $\mathrm{Cat}_\infty^{\mathrm{ex},1}$ denote the 1-category of all small stable ∞-categories and exact functors between them. Then the $S_\bullet$-construction gives rise to a functor of 1-categories

$$\mathrm{Cat}_\infty^{\mathrm{ex},1} \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{sSet}) \xrightarrow{|-|} \mathrm{sSet},$$

where the first functor sends a small stable ∞-category A to the simplicial Kan-complex $S_\bullet(A)$ and $|-|$ denotes the functor which sends a bisimplicial set to its diagonal. If now $f\colon A \to B$ is a Joyal equivalence of small stable ∞-categories, the induced map

$$S_n(f)\colon S_n(A) \to S_n(B)$$

is a homotopy equivalence and thus the induced morphism of diagonal simplicial sets is a weak equivalence by [GJ09, §IV Proposition 1.9]. It is not hard to see that $\mathrm{Cat}_\infty^{\mathrm{ex}}$ is equivalent to the localization of the 1-category $\mathrm{Cat}_\infty^{\mathrm{ex},1}$ at the Joyal equivalences. By the universal property of the localization, we thus get an induced functor

$$|S_\bullet(-)|\colon \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathcal{S}$$

of ∞-categories. Furthermore, by composing with the loop space functor we also get a functor

$$K\colon \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathcal{S}.$$

3.2. Exact Sequences

Remark 3.10. Recall that an ∞-category C is called idempotent complete if every idempotent $F\colon \text{Idem} \to C$ admits a colimit in C (see [Lur09, §4.4.5]).

If C is stable, then C is idempotent complete if and only if $\text{ho}(C)$ is an idempotent complete 1-category in the ordinary sense, by [Lur12, Lemma 1.2.4.6]. Furthermore, the inclusion $\text{Cat}_\infty^\vee \hookrightarrow \text{Cat}_\infty$ of the full subcategory of small idempotent complete ∞-categories into all small ∞-categories admits a left adjoint

$$(-)^{\text{idem}}\colon \text{Cat}_\infty \to \text{Cat}_\infty^\vee,$$

called *idempotent completion*. Moreover, it follows from [Lur09, Lemma 5.4.2.4] that, for any infinite regular cardinal κ we have

$$C^{\text{idem}} = \text{Ind}_\kappa(C)^{\kappa\text{-comp}},$$

where $(-)^{\kappa\text{-comp}}$ denotes the full subcategory spanned by the κ-compact objects. In particular, it follows from [Lur12, Proposition 1.1.3.6] that C^{idem} is again stable.

Notation 3.11. We write $\text{Cat}_\infty^{\text{perf}}$ for the full subcategory of $\text{Cat}_\infty^{\text{ex}}$ spanned by the idempotent complete small stable ∞-categories.

Definition 3.12. A sequence $A \xrightarrow{a} B \xrightarrow{b} C$ in $\text{Cat}_\infty^{\text{perf}}$ is called *exact* if

i) the morphism a is fully faithful,

ii) the composite $b \circ a$ is equivalent to the zero functor and

iii) the induced map $B/A \to C$ induces an equivalence $(B/A)^{\text{idem}} \xrightarrow{\simeq} C$, where B/A denotes the cofiber of a in $\text{Cat}_\infty^{\text{ex}}$.

Remark 3.13. It follows from Remark 3.10, that the adjunction above restricts to an adjunction

$$(-)^{\text{idem}}\colon \text{Cat}_\infty^{\text{ex}} \rightleftarrows \text{Cat}_\infty^{\text{perf}} :i.$$

If follows that condition iii) above can be rephrased by saying that the sequence $A \to B \to C$ is a cofiber sequence in the ∞-category $\text{Cat}_\infty^{\text{perf}}$.

Our next goal is to provide a more explicit description of the cofiber B/A in $\text{Cat}_\infty^{\text{ex}}$.

Definition 3.14. Let B be a stable ∞-category and let $A \subseteq B$ be the inclusion of a stable subcategory. Let $W_{B,A}$ denote the class of all morphisms f in B such that the cofiber of f is contained in the essential image of A.

Proposition 3.15. *Let $i\colon A \hookrightarrow B$ be a fully faithful exact functor between small idempotent complete stable ∞-categories. Then there is a canonical equivalence*

$$\varphi\colon B[W_{B,A}^{-1}] \xrightarrow{\simeq} B/A.$$

Proof: We will start by showing that $B[W_{B,A}^{-1}]$ is stable. It is easy to see that $W_{B,A}$ is stable under both pullbacks and pushouts and satisfies the 2-out-of-3 property. It follows that the triple $(B, W_{B,A}, B)$ is both an ∞-category with weak equivalences and fibrations and an ∞-category with weak equivalences and cofibrations in the sense of [Cis19, Definition 7.4.12]. It follows from [Cis19, Theorem 7.5.18] and its dual version that $B[W_{B,A}^{-1}]$ has all finite limits and colimits and the localization functor

$$\gamma\colon B \to B[W_{B,A}^{-1}]$$

preserves finite limits and colimits. It follows that $B[W_{B,A}^{-1}]$ is pointed. To show that it is stable, it suffices to see that, for every $x \in B[W_{B,A}^{-1}]$, the canonical morphisms

$$x \to \Sigma\Omega x \qquad \text{and} \qquad \Sigma\Omega x \to x$$

are equivalences, where Σ and Ω denote the suspension and the loop functor, respectively. By construction of the localization, the map γ is essentially surjective, so this follows because γ is exact and B is stable. We observe that, since the composite

$$A \hookrightarrow B \xrightarrow{\pi} B/A$$

is 0, it follows that π inverts all morphisms in $W_{B,A}$: A morphism in an stable ∞-category is an equivalence if and only if its cofiber is 0. Thus we get an induced functor $\varphi\colon B[W_{B,A}^{-1}] \to B/A$ by the universal property of the localization. Now we observe that, for any small stable ∞-category D, we have a pullback square

$$\begin{array}{ccc} \mathrm{Fun}^{\mathrm{ex}}(B/A, D)^{\simeq} & \longrightarrow & \mathrm{Fun}^{\mathrm{ex}}(B, D)^{\simeq} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Fun}^{\mathrm{ex}}(A, D)^{\simeq} \end{array}$$

which shows that $\mathrm{Fun}^{\mathrm{ex}}(B/A, D)^{\simeq}$ is given by the full subgroupoid of all functors $F\colon B \to D$ with $F \circ i \simeq 0$. Again, since a morphism in D is an equivalence if and only if its cofiber is 0, it follows that

$$\mathrm{Fun}^{\mathrm{ex}}(B/A, D)^{\simeq} \simeq \mathrm{Fun}^{\mathrm{ex}}_{W_{B,A}}(B, D)^{\simeq} \simeq \mathrm{Fun}^{\mathrm{ex}}(B[W_{A,B}^{-1}], D)^{\simeq},$$

where the last equivalence follows from [Cis19, Proposition 7.5.28]. Furthermore, it is easy to see that the above equivalence is induced by φ and the claim follows from the Yoneda lemma. □

Definition 3.16. For a fully faithful functor $i\colon A \to B$ in $\mathrm{Cat}_\infty^{\mathrm{ex}}$, we call the cofiber B/A the *Verdier-quotient* of i.

For later use, we will now also study exact sequences of stable presentable ∞-categories and their relationship to exact sequences of small stable ∞-categories.

Notation 3.17. We will write $\mathcal{P}r^L_{\mathrm{st}}$ for the full subcategory of $\mathcal{P}r^L$ spanned by the presentable stable ∞-categories.

Definition 3.18. A sequence $\mathcal{C} \xrightarrow{u} \mathcal{D} \xrightarrow{v} \mathcal{E}$ in $\mathcal{P}r^L_{\mathrm{st}}$ is called *exact* if

i) the composite $v \circ u$ is zero,

ii) the functor u is fully faithful and

iii) the sequence is a cofiber sequence in $\mathcal{P}r^L_{\mathrm{st}}$.

Proposition 3.19. *Let $A \xrightarrow{u} B \xrightarrow{v} C$ be an exact sequence in $\mathrm{Cat}^{\mathrm{perf}}_\infty$. Then the induced sequence*

$$\mathrm{Ind}(A) \xrightarrow{\mathrm{Ind}(u)} \mathrm{Ind}(B) \xrightarrow{\mathrm{Ind}(v)} \mathrm{Ind}(C)$$

is exact.

Proof: It follows from [Lur09, Proposition 5.3.5.11] that $\mathrm{Ind}(u)$ is fully faithful and it is clear that $\mathrm{Ind}(v) \circ \mathrm{Ind}(u) \simeq 0$. The fact that

$$\mathrm{Ind}(A) \xrightarrow{\mathrm{Ind}(u)} \mathrm{Ind}(B) \xrightarrow{\mathrm{Ind}(v)} \mathrm{Ind}(C)$$

is a cofiber sequence follows by combining Remark 3.13, [Lur09, Proposition 5.3.5.10] and [Lur09, Proposition 5.3.5.13]. □

Lemma 3.20. *Let $i\colon \mathcal{C} \hookrightarrow \mathcal{D}$ be a fully faithful functor in $\mathcal{P}r^L_{\mathrm{st}}$. Let κ be an infinite regular cardinal and assume that $\mathcal{C}$ is κ-compactly generated and that i preserves κ-compact objects. Then the canonical functor*

$$\mathcal{D}[W^{-1}_{\mathcal{D}^{\kappa\text{-comp}},\mathcal{C}^{\kappa\text{-comp}}}] \to \mathcal{D}[W^{-1}_{\mathcal{D},\mathcal{C}}]$$

is an equivalence.

Proof: Note that the collection of morphisms $W_{\mathcal{D}^{\kappa\text{-comp}},\mathcal{C}^{\kappa\text{-comp}}}$ is a (small) set. By [Lur09, Proposition 5.5.4.15] and [Lur09, Proposition 5.5.4.20], it thus suffices that the strongly saturated collections of morphisms generated by $W_{\mathcal{D}^{\kappa\text{-comp}},\mathcal{C}^{\kappa\text{-comp}}}$ and $W_{\mathcal{D},\mathcal{C}}$ agree (see [Lur09, Definition 5.5.4.5]). Since clearly $W_{\mathcal{D}^{\kappa\text{-comp}},\mathcal{C}^{\kappa\text{-comp}}} \subseteq W_{\mathcal{D},\mathcal{C}}$, it suffices to see that $W_{\mathcal{D},\mathcal{C}}$ is contained in the smallest strongly saturated class generated by $W_{\mathcal{D}^{\kappa\text{-comp}},\mathcal{C}^{\kappa\text{-comp}}}$, which we will denote by S. For this, let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be a cofiber sequence in $\mathcal{D}$ such that $Z \in i(\mathcal{C})$. As $\mathcal{C}$ is κ-compactly generated, we can find a κ-filtered diagram $Z_\bullet\colon I \to \mathcal{C}^{\kappa\text{-comp}}$ such that

$$\operatorname*{colim}_{i \in I} Z_i \simeq Z.$$

Let us write $Y_i = Y \times_Z Z_i$ and $X_i = \mathrm{fib}(Y_i \to Z_i)$. Let us consider the fiber T_i of the canonical map $f_i\colon X_i \to Y_i$. Then, in the diagram

$$\begin{array}{ccc} T_i & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & Y_i \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z_i \end{array}$$

all squares are cartesian. Since C is stable, the cofiber of the canonical map $T_i \to 0$ is equivalent to Z_i and thus contained in $C^{\kappa\text{-comp}}$. Furthermore, the morphism i preserves κ-compact objects by assumption and thus $T_i \simeq \Omega Z_i$ is contained in $\mathcal{D}^{\kappa\text{-comp}}$. Thus the map $T_i \to 0$ is contained in $W_{\mathcal{D}^{\kappa\text{-comp}}, C^{\kappa\text{-comp}}}$ and it follows that f_i is, as a pushout of a map in S, contained in S. Since by construction

$$X \xrightarrow{f} Y \simeq \operatorname*{colim}_{i \in I}(X_i \xrightarrow{f_i} Y_i),$$

it follows that f is contained in S and the claim follows. □

Construction 3.21. In the situation of Lemma 3.20, it follows that $\mathcal{D}[W_{\mathcal{D},C}{}^{-1}]$ is equivalent to the full subcategory of $\mathcal{D}$ spanned by the $W_{\mathcal{D},C}$-local objects. In particular, we get a fully faithful functor $\mathcal{D}[W_{\mathcal{D},C}{}^{-1}] \hookrightarrow \mathcal{D}$ that admits a left adjoint L. By [Lur09, Proposition 5.5.4.20], this adjoint L induces an equivalence

$$\mathrm{Fun}^L(\mathcal{D}[W_{\mathcal{D},C}{}^{-1}], \mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}^L_{W_{\mathcal{D},C}}(\mathcal{D}, \mathcal{E})$$

for any presentable ∞-category $\mathcal{E}$. Here, by $\mathrm{Fun}^L_{W_{\mathcal{D},C}}(\mathcal{D}, \mathcal{E})$ we denote the full subcategory of the ∞-category $\mathrm{Fun}^L(\mathcal{D}, \mathcal{E})$ spanned by all functors $F: \mathcal{D} \to \mathcal{E}$ that invert all morphisms in $W_{\mathcal{D},C}$.

Corollary 3.22. *Let $i: C \to \mathcal{D}$ be a fully faithful functor in $\mathcal{P}r^L_{\mathrm{st}}$. Let κ be an infinite regular cardinal and assume that C is κ-compactly generated and that i preserves κ-compact objects. Then there is a canonical equivalence*

$$\mathcal{D}[W_{\mathcal{D},C}{}^{-1}] \xrightarrow{\simeq} \mathcal{D}/C = \mathrm{cofib}(i).$$

Proof: By definition of the cofiber $\mathcal{D}/C$, we get a pullback square

$$\begin{array}{ccc} \mathrm{Fun}^L(\mathcal{D}/C, \mathcal{E})^{\simeq} & \longrightarrow & \mathrm{Fun}^L(\mathcal{D}, \mathcal{E})^{\simeq} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Fun}^L(C, \mathcal{E})^{\simeq} \end{array}$$

of spaces for any presentable stable ∞-category $\mathcal{E}$. But since a morphism in a stable ∞-category is an equivalence if and only if its cofiber is 0, it follows that the composite

$$\mathcal{D}[W_{\mathcal{D},C}{}^{-1}] \hookrightarrow \mathcal{D} \xrightarrow{\pi} \mathcal{D}/C$$

induces an equivalence

$$\mathrm{Fun}^L(\mathcal{D}/C, \mathcal{E})^{\simeq} \xrightarrow{\simeq} \mathrm{Fun}^L_{W_{\mathcal{D},C}}(\mathcal{D}, \mathcal{E})^{\simeq} \xrightarrow{\simeq} \mathrm{Fun}^L(\mathcal{D}[W_{\mathcal{D},C}{}^{-1}], \mathcal{E})^{\simeq}$$

for any presentable ∞-category $\mathcal{E}$. Now the claim follows from the Yoneda lemma if we can show that $\mathcal{D}[W_{\mathcal{D},C}{}^{-1}]$ is presentable and stable. Presentability is clear. To show that it is stable it suffices to see that the suspension functor $\Sigma: \mathcal{D} \to \mathcal{D}$ preserves $W_{\mathcal{D},C}$-local objects, by [Lur12, Lemma 1.1.3.3]. But this follows by adjunction, as the loop functor $\Omega: \mathcal{D} \to \mathcal{D}$ sends morphisms in $W_{\mathcal{D},C}$ to morphisms in $W_{\mathcal{D},C}$. This completes the proof. □

We are now able to provide a partial converse to Proposition 3.19.

Proposition 3.23. *Consider an exact sequence $\mathcal{C} \xrightarrow{u} \mathcal{D} \xrightarrow{v} \mathcal{E}$ of compactly generated presentable stable ∞-categories. Let $\kappa \geq \omega$ be a regular cardinal and assume that u and v both preserve κ-compact objects. Then the induced sequence*

$$\mathcal{C}^{\kappa\text{-comp}} \to \mathcal{D}^{\kappa\text{-comp}} \to \mathcal{E}^{\kappa\text{-comp}}$$

is an exact sequence in $\mathrm{Cat}_\infty^{\mathrm{perf}}$.

Proof: We would like to show that the induced functor

$$\varphi\colon \mathcal{D}^{\kappa\text{-comp}}/\mathcal{C}^{\kappa\text{-comp}} \to \mathcal{E}^{\kappa\text{-comp}}$$

is an equivalence after idempotent completion. Note that it follows from [Lur09, Proposition 5.4.2.9] and [Lur09, Remark A.2.6.4] that $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ are κ-accessible. Now let $\mathcal{A}$ be any κ-accessible ∞-category. Then we compute

$$\begin{aligned}\mathrm{Fun}^{\kappa\text{-acc}}(\mathrm{Ind}_\kappa(\mathcal{D}^{\kappa\text{-comp}}/\mathcal{C}^{\kappa\text{-comp}}), \mathcal{A}) &\simeq \mathrm{Fun}(\mathcal{D}^{\kappa\text{-comp}}/\mathcal{C}^{\kappa\text{-comp}}, \mathcal{A}) \\ &\simeq \mathrm{Fun}_{W_{\mathcal{D}^{\kappa\text{-comp}},\mathcal{C}^{\kappa\text{-comp}}}}(\mathcal{D}^{\kappa\text{-comp}}, \mathcal{A}) \\ &\simeq \mathrm{Fun}^{\kappa\text{-acc}}_{W_{\mathcal{D}^{\kappa\text{-comp}},\mathcal{C}^{\kappa\text{-comp}}}}(\mathrm{Ind}_\kappa(\mathcal{D}^{\kappa\text{-comp}}), \mathcal{A}) \\ &\simeq \mathrm{Fun}^{\kappa\text{-acc}}(\mathcal{E}, \mathcal{A}),\end{aligned}$$

where the first and the third equivalence follow from [Lur09, Proposition 5.3.5.10], the second equivalence from Proposition 3.15 and the last equivalence from Lemma 3.20. Furthermore, it is easy to see that the above equivalence is induced by the functor

$$\mathrm{Ind}_\kappa(\varphi)\colon \mathrm{Ind}_\kappa(\mathcal{D}^{\kappa\text{-comp}}/\mathcal{C}^{\kappa\text{-comp}}) \to \mathrm{Ind}_\kappa(\mathcal{E}^{\kappa\text{-comp}}) \simeq \mathcal{E},$$

which therefore is an equivalence, too. By restricting this to κ-compact objects again, the claim follows from [Lur09, Lemma 5.4.2.4]. □

Corollary 3.24 (Thomason-Neeman localization theorem)**.** *Let us consider an exact sequence $\mathcal{C} \xrightarrow{u} \mathcal{D} \xrightarrow{v} \mathcal{E}$ of compactly generated presentable stable ∞-categories. Let $\kappa \geq \omega$ be a regular cardinal and assume that u and v both preserve κ-compact objects. Let $e \in \mathcal{E}^{\kappa\text{-comp}}$. Then there is a κ-compact object $d \in \mathcal{D}^{\kappa\text{-comp}}$ such that e is a retract of $v(d)$.*

Proof: This follows from Proposition 3.23 because the functor $\mathcal{D}^{\kappa\text{-comp}} \to \mathcal{D}^{\kappa\text{-comp}}/\mathcal{C}^{\kappa\text{-comp}}$ is, as a localization, essentially surjective. □

In fact, we can be a bit more specific about the object d above. For this we will use the following easy result about K_0:

Proposition 3.25. *Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full stable subcategory such that every object in $\mathcal{C}$ is a retract of an object in $\mathcal{C}_0$. Then the induced map $i\colon K_0(\mathcal{C}_0) \to K_0(\mathcal{C})$ is injective and an object X is in the essential image of $\mathcal{C}_0$ if and only if $[X]$ is in the image of i.*

Proof: See [Lur14, Lecture 14, Proposition 19]. □

It follows that, in the above situation, the object $X \oplus \Sigma X$ is always contained in the essential image of C_0. Thus we obtain the following strengthened version of Corollary 3.24:

Corollary 3.26. *Let us consider an exact sequence $C \xrightarrow{u} \mathcal{D} \xrightarrow{v} \mathcal{E}$ of compactly generated presentable stable ∞-categories. Let $\kappa \geq \omega$ be a regular cardinal and assume that u and v both preserve κ-compact objects. Let $e \in \mathcal{E}^{\kappa\text{-comp}}$. Then there is a $d \in \mathcal{D}^{\kappa\text{-comp}}$ such that*

$$v(d) \simeq e \oplus \Sigma e.$$

3.3. The Waldhausen Fibration Theorem

This section is devoted to proving our version of the Waldhausen Fibration Theorem (see Theorem 3.41) and then deduce that K-theory takes exact sequences of small stable ∞-categories to fiber sequences.

The main ingredient we will use is the following:

Theorem 3.27 (Additivity)**.** *Let C be a stable ∞-category. Then the canonical morphism*

$$\begin{aligned} \mathrm{Fun}(\Delta^1, C) &\to C \times C \\ (f\colon A \to B) &\mapsto (A, \mathrm{cofib}(f)) \end{aligned}$$

induces an equivalence

$$|S_\bullet(\mathrm{Fun}(\Delta^1, C))| \to |S_\bullet(C \times C)| \simeq |S_\bullet(C)| \times |S_\bullet(C)|.$$

Proof: This is originally due to Waldhausen, see [Wal85, Theorem 1.4.2]. See the proof of [Lur14, Lecture 17, Theorem 1] for an argument using the language of this thesis. □

Let us now collect a few corollaries:

Corollary 3.28. *Consider the three functors $G, G', G''\colon \mathrm{Fun}(\Delta^1, C) \to C$ given by*

$$\begin{aligned} G(A \to B) &= A, \\ G'(A \to B) &= B \\ \textit{and } G''(A \to B) &= \mathrm{cofib}(A \to B). \end{aligned}$$

Then $|S_\bullet(G)| + |S_\bullet(G'')|$ is homotopic to $|S_\bullet(G')|$.

Proof: We observe that the functor

$$\begin{aligned} \Phi\colon C \times C &\to \mathrm{Fun}(\Delta^1, C) \\ (A, B) &\mapsto (A \to A \amalg B) \end{aligned}$$

is a right inverse to the functor in Theorem 3.27. Therefore it induces an equivalence after applying $|S_\bullet(-)|$. So the claim follows since $G' \circ \Phi \simeq (G \amalg G'') \circ \Phi$. □

Corollary 3.29. *Let $\mathcal{C}$ and $\mathcal{D}$ be stable ∞-categories and let*

$$F \xrightarrow{a} F' \to F''$$

be a cofiber sequence in $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$*. Then one has a homotopy equivalence*

$$|S_\bullet(F')| \simeq |S_\bullet(F)| + |S_\bullet(F'')|.$$

Proof: The natural transformation a determines an exact functor $H\colon \mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{D})$ such that, with the notation from above,

$$\begin{aligned} G \circ H &= F \\ G' \circ H &= F', \\ \text{and } G'' \circ H &= F''. \end{aligned}$$

So the claim follows from the above corollary. □

Example 3.30. Let $\mathcal{C}$ and $\mathcal{D}$ be stable ∞-categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor. Then the cofiber sequence of endofunctors

$$F \to 0 \to \Sigma_{\mathcal{D}} \circ F,$$

where $\Sigma_{\mathcal{D}}$ is the suspension functor, shows that there is a homotopy

$$0 \simeq |S_\bullet(F)| + |S_\bullet(\Sigma F)|.$$

We now turn towards the proof of the Fibration Theorem. For this, we will need a few technical preliminaries:

Lemma 3.31. *Let $\mathcal{C}$ be an ∞-category and let $W \subseteq \mathcal{C}$ be a subcategory. Assume that there is a right calculus of fractions of W in $\mathcal{C}$ (in the sense of [Cis19, Definition 7.2.6]). Then W is saturated if and only if it satisfies the 2-out-of-6 property.*

Proof: Consider the localization functor $\gamma\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$. Recall that W is called saturated if, for every morphism s in $\mathcal{C}$, the morphism $\gamma(s)$ is an equivalence if and only if $s \in W$. Since a morphism is an equivalence if and only if it becomes an isomorphism in the homotopy category and since localizations are compatible with taking homotopy categories, we may reduce to the 1-categorical case by [Cis19, Corollary 7.2.12]. In the 1-categorical case this is well known, see for example [KS06, Proposition 7.1.20]. □

Example 3.32. Let $\mathcal{C}$ be an ∞-category with finite limits and let W be a subcategory. If W is closed under pullbacks, then there exists a right calculus of fractions of W in $\mathcal{C}$ by [Cis19, Proposition 7.2.16]. Thus, if W also satisfies 2-out-of-6, the above Proposition shows that W is saturated.

Proposition 3.33. *Let B be a stable ∞-category and let $A \subseteq B$ be the inclusion of a full stable subcategory that is closed under retracts in B. Then $W_{B,A}$ is saturated.*

Proof: By the above example, it suffices to see that $W_{B,A}$ is stable under pullbacks and satisfies 2-out-of-6. Stability under pullbacks is obvious. So consider three composable morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$$

such that gf and hg are in W. We get a diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & Y/X & \longrightarrow & Z/X & \longrightarrow & T/X \\ & & \downarrow & & \downarrow & & \downarrow \\ & & * & \longrightarrow & Z/Y & \longrightarrow & T/Y \end{array}$$

where all squares are pushout and pullback squares. The bottom right square gives rise to the pullback square

$$\begin{array}{ccc} \Omega T/Y & \longrightarrow & Z/X \\ \downarrow & & \downarrow \\ * & \longrightarrow & T/X \oplus Z/Y \end{array}$$

Since by assumption Z/X and $\Omega T/Y$ are in the essential image of A, the cofiber $T/X \oplus Z/Y$ is also in the essential image of A. But now both T/X and Z/Y are rectracts of $T/X \oplus Z/Y$ and thus by assumption in the essential image of A. □

To simplify notation a bit, we will from now on write $\mathrm{Gap}_n(B)$ instead of $\mathrm{Gap}_{[n]}(B)$.

Construction 3.34. Let B be a stable ∞-category. We define $P(\mathrm{Gap}_\bullet(B))$ to be the simplicial ∞-category given by precomposing $\mathrm{Gap}_\bullet(B)$ with the functor

$$s\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$$

that takes $[n]$ to $[n+1]$ and a morphism $\alpha\colon [m] \to [k]$ to

$$\begin{aligned} s(\alpha)\colon [m+1] &\to [k+1] \\ i &\mapsto \begin{cases} 0 & \text{if } i = 0 \\ \alpha(i) & \text{else.} \end{cases} \end{aligned}$$

Now let $i\colon A \hookrightarrow B$ be the inclusion of a full stable subcategory. We define an ∞-category $\mathrm{Gap}_n(B, A)$ via the pullback square

$$\begin{array}{ccc} \mathrm{Gap}_n(B,A) & \longrightarrow & \mathrm{Gap}_{n+1}(B) \\ \downarrow{\scriptstyle \delta_0'} & & \downarrow{\scriptstyle \delta_0} \\ \mathrm{Gap}_n(A) & \xrightarrow{\mathrm{Gap}_n(i)} & \mathrm{Gap}_n(B) \end{array}$$

where δ_0 is the map induced by the inclusion $\{1, \ldots, n+1\} \hookrightarrow \{0, \ldots, n+1\}$. Note that the map δ_0 is compatible with the face and degeneracy maps and thus assembles to a morphism of simplicial ∞-categories. We can thus define a new simplicial ∞-category $\mathrm{Gap}_\bullet(B, A)$ by the pullback

$$\begin{array}{ccc} \mathrm{Gap}_\bullet(B,A) & \longrightarrow & P(\mathrm{Gap}_\bullet(B)) \\ \downarrow & & \downarrow \\ \mathrm{Gap}_\bullet(A) & \longrightarrow & \mathrm{Gap}_\bullet(B) \end{array}$$

Remark 3.35. Note that the equivalence of Lemma 3.7 restricts to an equivalence

$$\mathrm{Gap}_n(B,A) \xrightarrow{\simeq} \mathrm{Fun}(\Delta^n, B)_{W_{B,A}},$$

where $\mathrm{Fun}(\Delta^n, B)_{W_{B,A}} \subseteq \mathrm{Fun}(\Delta^n, B)$ is the full subcategory spanned by those functors $\Delta^n \to B$ where all morphisms $\Delta^{\{i,i+1\}} \hookrightarrow \Delta^n \to B$ lie in $W_{B,A}$. Furthermore, we observe that this equivalence is in fact compatible with the canonical face and degeneracy maps. Thus we get an equivalence

$$\mathrm{Gap}_\bullet(B,A) \xrightarrow{\simeq} \mathrm{Fun}(\Delta^\bullet, B)_{W_{B,A}}$$

of simplicial ∞-categories induced by restriction.

Notation 3.36. Given a simplicial simplicial set $U_\bullet$, we will from now on write $|U_\bullet|$ for the associated diagonal simplicial set. Similarly, when given a bisimplicial simplicial set $X_{\bullet,\bullet}$, we will write $|X_{\bullet,\bullet}|$ for the *geometric realization* of $X_{\bullet,\bullet}$, i.e. for the diagonal simplicial set of the associated trisimplicial set.

Construction 3.37. Note that precomposing with the functor $\Delta^n \to \Delta^0$ induces a morphism

$$i_B \colon B \to \mathrm{Fun}(\Delta^n, B)_{W_{B,A}} \simeq \mathrm{Gap}_n(B,A).$$

Considering B as a constant simplicial ∞-category, we get a sequence of simplicial ∞-categories

$$B \to \mathrm{Gap}_\bullet(B,A) \to \mathrm{Gap}_\bullet(A).$$

Finally, this induces a sequence of bisimplicial Kan-complexes

$$S_\bullet(B) \to S_\bullet(\mathrm{Gap}_\bullet(B,A)) \to S_\bullet(\mathrm{Gap}_\bullet(A)). \tag{1}$$

We now get the following version of [Wal85, Proposition 1.5.5]:

Proposition 3.38. *Let B be a stable ∞-category and let $A \subseteq B$ be a full stable subcategory. Then the induced sequence*

$$|S_\bullet(B)| \to |S_\bullet(\mathrm{Gap}_\bullet(B,A))| \to |S_\bullet(\mathrm{Gap}_\bullet(A))|$$

is a homotopy fiber sequence.

Proof: The proof is a direct adaption of the argument given in [Wal85]: Let us consider the simplicial simplicial sets

$$\begin{aligned} U_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} &\to \mathrm{sSet} \\ [n] &\mapsto |S_\bullet \operatorname{Gap}_n(B,A)| \end{aligned}$$

and

$$\begin{aligned} V_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} &\to \mathrm{sSet} \\ [n] &\mapsto |S_\bullet \operatorname{Gap}_n(A)|. \end{aligned}$$

Then the sequence (1) from Construction 3.37 yields a sequence of simplicial simplicial sets

$$S_\bullet(B) \to U_\bullet \to V_\bullet$$

and, by [Wal78, Lemma 5.2], as $|S_\bullet \operatorname{Gap}_n(A)|$ is connected, it suffices to see that

$$|S_\bullet(B)| \to |S_\bullet \operatorname{Gap}_n(B,A)| \to |S_\bullet \operatorname{Gap}_n(A)|$$

is a fiber sequence for all n. For this, we will construct a homotopy commutative diagram

$$\begin{array}{ccccc} |S_\bullet(B)| & \xrightarrow{(\mathrm{id},*)} & |S_\bullet(B)| \times |S_\bullet \operatorname{Gap}_n(A)| & \xrightarrow{\mathrm{pr}_2} & |S_\bullet \operatorname{Gap}_n(A)| \\ \uparrow & & \uparrow{\scriptstyle\varphi} & & \uparrow \\ |S_\bullet(B)| & \longrightarrow & |S_\bullet \operatorname{Gap}_n(B,A)| & \longrightarrow & |S_\bullet \operatorname{Gap}_n(A)| \end{array}$$

and show that the vertical morphism in the middle is a homotopy equivalence. To do so, we consider the functor

$$p\colon \operatorname{Gap}_n(B,A) \to B$$

induced by the inclusion $\{0\} \to [n]^{(2)}$ onto the element $(0,1)$ and the functor

$$\delta_0'\colon \operatorname{Gap}_n(B,A) \to \operatorname{Gap}_n(A)$$

from Construction 3.34. Then we set $\varphi = (F, \delta_0')$, which by construction makes the above diagram commute. We now consider the functor

$$i_B\colon B \to \operatorname{Gap}_n(B,A)$$

from Construction 3.37 and the functor

$$G\colon \operatorname{Gap}_n(A) \to \operatorname{Gap}_n(B,A).$$

The latter corresponds, under the identification of Remark 3.35, to the canonical functor

$$\operatorname{Fun}(\Delta^{n-1}, A) \to \operatorname{Fun}(\Delta^n, W_{B,A}),$$

that sends an n-simplex $\alpha\colon \Delta^n \to A$ in A to the essentially unique $n+1$-simplex

$$G(\alpha)\colon \Delta^{n+1} \to \mathcal{W}_{B,A},$$

that is determined by $G(\alpha)|_{\Delta^{\{1,\dots,n+1\}}} = \alpha$ and $G(\alpha)|_{\Delta^{\{0\}}} \simeq 0$. We now observe that we have canonical natural transformations

$$i_B \circ F \to \mathrm{id}_{\mathrm{Gap}_n(B,A)} \qquad \text{and} \qquad \mathrm{id}_{\mathrm{Gap}_n(B,A)} \to G \circ \delta_0',$$

that give rise to a cofiber sequence

$$i_B \circ F \to \mathrm{id}_{\mathrm{Gap}_n(B,A)} \to G \circ \delta_0'$$

of endofunctors of B. By Corollary 3.29, it follows that the functor $|S_\bullet(i_B \amalg G)|$ is a left inverse of $|S_\bullet(\varphi)|$. It is furthermore clear that $\varphi \circ (i_B \amalg G) \simeq \mathrm{id}_{B\times \mathrm{Gap}_n(A)}$, which completes the proof. □

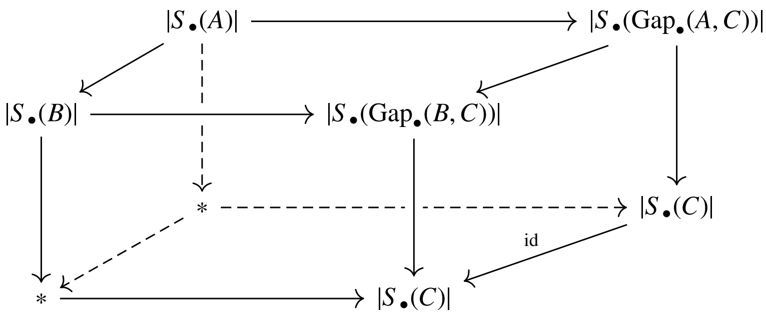
Corollary 3.39. *Let B be a stable ∞-category and let $A, C \subseteq B$ be full stable subcategories with $C \subseteq A$. Then we get a homotopy pullback square*

$$\begin{array}{ccc} |S_\bullet(A)| & \longrightarrow & |S_\bullet(\mathrm{Gap}_\bullet(A,C))| \\ \downarrow & & \downarrow \\ |S_\bullet(B)| & \longrightarrow & |S_\bullet(\mathrm{Gap}_\bullet(B,C))| \end{array}$$

Here, the right horizontal map is induced by the commutative diagram

$$\begin{array}{ccccc} \mathrm{Gap}_\bullet(C) & \longrightarrow & \mathrm{Gap}_\bullet(A) & \longleftarrow & P(\mathrm{Gap}_\bullet(A)) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow \\ \mathrm{Gap}_\bullet(C) & \longrightarrow & \mathrm{Gap}_\bullet(B) & \longleftarrow & P(\mathrm{Gap}_\bullet(B)) \end{array}$$

Proof: We have a commutative cube



where the front and back face are homotopy cartesian by the proposition above. Since the bottom square is obviously homotopy cartesian, it follows that the top square is a homotopy pullback square, as desired. □

The main input for the proof of the Fibration Theorem is the following:

Proposition 3.40. *Let $A \hookrightarrow B$ be a fully faithful functor in* $\mathrm{Cat}_\infty^{\mathrm{perf}}$. *Let $p\colon B \to B/A$ be the canonical functor into the Verdier-Quotient. Then there is a homotopy equivalence*

$$\Phi\colon |S_\bullet(\mathrm{Gap}_\bullet(B,A))| \to |S_\bullet(B/A)|$$

which makes the diagram

$$\begin{array}{ccc} |S_\bullet(B)| & & \\ \downarrow & \overset{|S_\bullet(p)|}{\searrow} & \\ |S_\bullet(\mathrm{Gap}_\bullet(B,A))| & \xrightarrow{\Phi} & |S_\bullet(B/A)| \end{array}$$

commute. Here, the map $|S_\bullet(B)| \to |S_\bullet(\mathrm{Gap}_\bullet(B,A))|$ is the one from Construction 3.37.

Proof: To abbreviate notation, let us define the trisimplicial sets

$$\begin{aligned} \mathcal{A}_{\bullet,\bullet,\bullet}\colon \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} &\to \mathrm{Set} \\ ([m],[n],[k]) &\mapsto S_m(\mathrm{Fun}(\Delta^n, B)_{W_{B,A}})_k \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_{\bullet,\bullet,\bullet}\colon \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} &\to \mathrm{Set} \\ ([m],[n],[k]) &\mapsto \mathrm{Gap}_m^{W_{B,A}}(B)_n. \end{aligned}$$

Here, we define $\mathrm{Gap}_m^{W_{B,A}}(B)$ to be the subcategory of $\mathrm{Gap}_m(B)$ on all objects and those natural transformations whose components lie in $W_{B,A}$. We furthermore observe that we have a canonical identification

$$\mathrm{Gap}_m^{W_{B,A}}(B)_n = S_m(\mathrm{Fun}(\Delta^n, B)_{W_{B,A}})_0$$

as subsets of $\mathrm{Hom}_{\mathrm{sSet}}([m]^{(2)}\times\Delta^n, B)$. Thus we get a canonical morphism $\varphi_{\bullet,\bullet,\bullet}\colon \mathcal{B}_{\bullet,\bullet,\bullet} \hookrightarrow \mathcal{A}_{\bullet,\bullet,\bullet}$ induced by the degeneracy maps. We would like to show that the induced map of diagonal simplicial sets $|\varphi_{\bullet,\bullet,\bullet}|$ is a weak equivalence. By [GJ09, §IV Proposition 1.9], it suffices to see that, for all $m, k \in \mathbb{N}$, the map of simplicial sets

$$\varphi_{m,\bullet,k}\colon \mathcal{B}_{m,\bullet,k} \to \mathcal{A}_{m,\bullet,k}$$

is a weak equivalence. We note that by construction $\varphi_{m,\bullet,k}$ fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{B}_{m,\bullet,k} & \xrightarrow{\varphi_{m,\bullet,k}} & \mathcal{A}_{m,\bullet,k} \\ \| & & \| \\ \mathrm{Gap}_m^{W_{B,A}}(B) & \longrightarrow & S_m(\mathrm{Fun}(\Delta^\bullet, B)_{W_{B,A}})_k \\ \downarrow & & \downarrow \\ \mathrm{Fun}([m]^{(2)}, B) & \xrightarrow{\psi} & \mathrm{Fun}([m]^{(2)} \times \Delta^k, B) \end{array}$$

where ψ is the functor induced by the projection $[m]^{(2)} \times \Delta^k \to [m]^{(2)}$. Now we observe that ψ has a left inverse $\xi\colon \mathrm{Fun}([m]^{(2)} \times \Delta^k, B) \to \mathrm{Fun}([m]^{(2)}, B)$, induced by the inclusion

$$i\colon [m]^{(2)} \times \{0\} \to [m]^{(2)} \times \Delta^k.$$

Furthermore, there is a canonical homotopy

$$K\colon \Delta^1 \times [m]^{(2)} \times \Delta^k \to [m]^{(2)} \times \Delta^k$$

from the identity to the composite $[m]^{(2)} \times \Delta^k \to [m]^{(2)} \times \{0\} \xrightarrow{i} [m]^{(2)} \times \Delta^k$, which then induces a homotopy

$$H\colon \Delta^1 \times \mathrm{Fun}([m]^{(2)} \times \Delta^k, B) \to \mathrm{Fun}([m]^{(2)} \times \Delta^k, B)$$

between $\psi \circ \xi$ and the identity. It is now easy to check that ξ restricts to a map

$$\mathcal{A}_{m,\bullet,k} \to \mathcal{B}_{m,\bullet,k}$$

and H restricts to a map

$$\Delta^1 \times \mathcal{A}_{m,\bullet,k} \to \mathcal{A}_{m,\bullet,k}$$

and thus $\varphi_{m,\bullet,k}$ is a homotopy equivalence, as desired.

Now let us consider the morphism

$$\mathrm{diag}\colon |\mathcal{A}_{\bullet,\bullet,\bullet}| \to |\mathcal{B}_{\bullet,\bullet,\bullet}|,$$

for every $n \in \mathbb{N}$ given by the map of sets

$$\mathrm{diag}_n\colon S_n(\mathrm{Fun}(\Delta^n, B)_{W_{B,A}})_n \to S_n(\mathrm{Fun}(\Delta^n, B)_{W_{B,A}})_0 = \mathrm{Gap}_m^{W_{B,A}}(B)_n,$$

which is induced by the diagonal map $[n] \times \{0\} \to [n] \times [n]$. Observe that this is a right inverse of $|\varphi_{\bullet,\bullet,\bullet}|$. In particular, the morphism diag is a weak equivalence as well.

Note that by definition $|B_{\bullet,\bullet,\bullet}|$ is nothing but the diagonal of the bisimplicial set associated to the simplicial ∞-category $\mathrm{Gap}_\bullet^{W_{B,A}}(B)$. We observe that the localization functor

$$\gamma\colon B \to B[W_{B,A}^{-1}] = B/A$$

induces a map of bisimplicial ∞-categories

$$\mathrm{Gap}_\bullet^{W_{B,A}}(B) \to S_\bullet(B/A). \tag{1}$$

We would like to show that this induces a weak equivalence after applying geometric realization. Again by [GJ09, §IV Proposition 1.9], it suffices to see that the induced map

$$\mathrm{Gap}_n^{W_{B,A}}(B) \to S_n(B/A)$$

is a weak equivalence for all n. For this, we observe that the equivalence from Lemma 3.7 restricts to an equivalence

$$\mathrm{Gap}_n^{W_{B,A}}(B) \xrightarrow{\simeq} \mathrm{Fun}^{W_{B,A}}(\Delta^n, B).$$

Here, we denote by $\mathrm{Fun}^{W_{B,A}}(\Delta^n, B)$ the subcategory of $\mathrm{Fun}(\Delta^n, B)$ on all objects and those natural transformations whose components are morphisms in $W_{B,A}$. Note that, since W is stable under pullbacks, it follows that $(B, W_{B,A}, B)$ is a category with weak equivalences and fibrations in the sense of [Cis19, Definition 7.4.12]. Thus it follows from [Cis19, Theorem 7.6.17] that the canonical functor

$$\mathrm{Fun}(\Delta^n, B)[\mathrm{Fun}^{W_{B,A}}(\Delta^n, B)^{-1}] \to \mathrm{Fun}(\Delta^n, B[W_{B,A}^{-1}])$$

is an equivalence. Note that by assumption A is closed under retracts in B and thus the subcategory $W_{B,A} \subseteq B$ is saturated by Proposition 3.33. It follows that the subcategory $\mathrm{Fun}^{W_{B,A}}(\Delta^n, B) \subseteq \mathrm{Fun}(\Delta^n, B)$ is saturated as well. Hence the canonical functor

$$\mathrm{Fun}^{W_{B,A}}(\Delta^n, B) \to \mathrm{Fun}(\Delta^n, B[W_{B,A}^{-1}])^{\simeq} \simeq S_n(B[W_{B,A}^{-1}])$$

is a weak homotopy equivalence by [Cis19, Corollary 7.6.9] and (1) indeed induces a weak equivalence after applying geometric realization.

So, putting all this together, we have constructed a homotopy equivalence

$$\Phi\colon |S_\bullet(\mathrm{Gap}_\bullet(B, A))| \simeq |B_{\bullet,\bullet,\bullet}| \xrightarrow{\mathrm{diag}} |A_{\bullet,\bullet,\bullet}| = |\mathrm{Gap}_\bullet^{W_{B,A}}(B)| \xrightarrow{\simeq} |S_\bullet(B/A)|,$$

as desired. Furthermore, it is now easy to see that the explicit map Φ constructed above makes the diagram

$$\begin{array}{ccc} |S_\bullet(B)| & & \\ \downarrow & \searrow^{|S_\bullet(p)|} & \\ |S_\bullet(\mathrm{Gap}_\bullet(B, A))| & \xrightarrow{\Phi} & |S_\bullet(B/A)| \end{array}$$

commute, which completes the proof. □

We are now finally able to deduce our version of Waldhausen's Fibration Theorem [Wal78, Theorem 1.6.4]:

Theorem 3.41. *Let $A \hookrightarrow B$ be a fully faithful functor in $\mathrm{Cat}_\infty^{\mathrm{perf}}$. Then the induced sequence*

$$|S_\bullet(A)| \to |S_\bullet(B)| \to |S_\bullet(B/A)|$$

is a homotopy fiber sequence.

Proof: We apply Corollary 3.39 to the sequence of inclusions $A \xrightarrow{\mathrm{id}} A \hookrightarrow B$ and thus get a homotopy cartesian square

$$\begin{array}{ccc} |S_\bullet(A)| & \longrightarrow & |S_\bullet(\mathrm{Gap}_\bullet(A, A))| \\ \downarrow & & \downarrow \\ |S_\bullet(B)| & \longrightarrow & |S_\bullet(\mathrm{Gap}_\bullet(B, A))| \end{array}$$

It follows that, by composing with the map Φ from Proposition 3.40, we get a homotopy cartesian square

$$\begin{array}{ccc} |S_\bullet(A)| & \longrightarrow & |S_\bullet(\mathrm{Gap}_\bullet(A,A))| \\ \downarrow & & \downarrow \\ |S_\bullet(B)| & \longrightarrow & |S_\bullet(B/A)| \end{array}$$

Moreover, we have $|S_\bullet(\mathrm{Gap}_\bullet(A,A))| \simeq |S_\bullet(A/A)|$, again by Proposition 3.40. Since $A/A \simeq *$, the claim follows. □

We would now like to apply this result to exact sequences in $\mathrm{Cat}_\infty^{\mathrm{perf}}$, for which we will need one more important theorem:

Theorem 3.42. *Let B be a stable ∞-category and let $B' \subseteq B$ be a full stable subcategory such that every object in B is a retract of an object in B'. Then there is a homotopy pullback square*

$$\begin{array}{ccc} K(B') & \longrightarrow & K(B) \\ \downarrow & & \downarrow \\ K_0(B') & \longrightarrow & K_0(B) \end{array}$$

Proof: See [Lur14, Lecture 18, Proposition 1]. □

Combining the last two theorems, we finally get the following result:

Theorem 3.43. *Let $A \to B \to C$ be an exact sequence in $\mathrm{Cat}_\infty^{\mathrm{perf}}$. Then the induced sequence*

$$K(A) \to K(B) \to K(C)$$

is a homotopy fiber sequence.

Proof: By Proposition 3.41, we have a homotopy fiber sequence

$$|S_\bullet(A)| \to |S_\bullet(B)| \to |S_\bullet(B/A)|.$$

The commutative diagram

$$\begin{array}{ccccc} |S_\bullet(A)| & \longrightarrow & |S_\bullet(B)| & \longrightarrow & |S_\bullet(B/A)| \\ \downarrow{\scriptstyle \varphi} & & \downarrow & & \downarrow \\ \mathrm{hofib}(|S_\bullet(g)|) & \longrightarrow & |S_\bullet(B)| & \xrightarrow{|S_\bullet(g)|} & |S_\bullet(C)| \end{array}$$

induces a commutative diagram of long exact sequences of homotopy groups. By assumption, we have that the induced map $(B/A)^{\mathrm{idem}} \to C$ is an equivalence. Thus we can regard B/A as a full subcategory of C such that every object in C is a retract of an object in B/A. It follows from Theorem 3.42 that in the diagram

$$\begin{array}{ccccccccc} K_{i+1}(B) & \longrightarrow & K_{i+1}(B/A) & \longrightarrow & K_i(A) & \longrightarrow & K_i(B) & \longrightarrow & K_i(B/A) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \pi_{i+1}(\varphi)} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ K_{i+1}(B) & \longrightarrow & K_{i+1}(C) & \longrightarrow & \pi_{i+1}(\mathrm{hofib}(|S_\bullet(g)|)) & \longrightarrow & K_i(B) & \longrightarrow & K_i(C) \end{array}$$

all vertical arrows except the middle one are isomorphisms, if $i \geq 1$. Therefore the middle vertical map is an isomorphism for all $i \geq 1$ as well.

If $i = 0$, the two left vertical maps are still bijective and the rightmost map is still injective by Proposition 3.25. Now, the refined version of the 5-lemma tells us that the middle vertical map is an isomorphism.

It follows that $\pi_i(\varphi)$ is an isomorphism for all $i \geq 1$. By Whitehead's theorem, the canonical map $\Omega(\varphi)\colon K(A) \to \Omega\,\mathrm{hofib}(|S_\bullet(g)|) \simeq \mathrm{hofib}(K(g))$ is a homotopy equivalence and we get the claim. □

3.4. Non-Connective K-Theory

In this section we will use the results developed in the last two sections in order to construct a localizing functor $\mathbb{K}\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$, such that $\Omega^\infty \circ \mathbb{K} \simeq K$. Here, we denote by Sp the ∞-category of spectra and by Ω^∞ the infinite loop space functor. We will call $\mathbb{K}$ the *non-connective K-theory* functor. We will roughly follow [BGT13, §9].

Definition 3.44. A functor

$$E\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \to T$$

to a stable ∞-category T is called *localizing*, if it takes exact sequences to fiber sequences in T.

Construction 3.45. Let κ be a regular uncountable cardinal. Consider the functor

$$\mathtt{F}\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Cat}_\infty^{\mathrm{perf}}$$

given by sending A to $\mathrm{Ind}(A)^{\kappa\text{-comp}}$. Note that, for any $A \in \mathrm{Cat}_\infty^{\mathrm{perf}}$, there is a canonical functor $A \to \mathrm{Ind}(A)^{\kappa\text{-comp}}$. We will denote its cofiber in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ by $\mathtt{E}(A)$. This construction assembles to a functor

$$\mathtt{E}\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Cat}_\infty^{\mathrm{perf}}.$$

By construction, we get a sequence of natural transformations

$$\mathrm{id}_{\mathrm{Cat}_\infty^{\mathrm{perf}}} \to \mathtt{F} \to \mathtt{E}$$

such that, for any $A \in \mathrm{Cat}_\infty^{\mathrm{perf}}$, the sequence

$$A \to \mathtt{F}(A) \to \mathtt{E}(A)$$

is an exact sequence in $\mathrm{Cat}_\infty^{\mathrm{perf}}$.

Lemma 3.46. *The functors* $\mathtt{F}$ *and* $\mathtt{E}$ *preserve exact sequences.*

Proof: The functor $\mathtt{F}$ preserves exact sequences by Proposition 3.19 and Proposition 3.23.

Thus it is clear that, for an exact sequence

$$A \xrightarrow{u} B \xrightarrow{v} C,$$

the induced sequence $\mathrm{E}(A) \xrightarrow{\mathrm{E}(u)} \mathrm{E}(B) \xrightarrow{\mathrm{E}(v)} \mathrm{E}(C)$ is still a cofiber sequence.

It remains to see that $\mathrm{E}(u)$ is fully faithful. We recall that the classes $W_{\mathrm{F}(A),A}$ and $W_{\mathrm{F}(B),B}$ are closed under pushouts. For $x \in \mathrm{F}(A)$, let us denote by $W_{\mathrm{F}(A),A}(x)$ the full subcategory of the slice $\mathrm{F}(A)_{x/}$ of all arrows that lie in $W_{\mathrm{F}(A),A}$. Analogously we define $W_{\mathrm{F}(B),B}(y)$ for $y \in \mathrm{F}(B)$. Then it follows from [Cis19, Theorem 7.2.8] and [Cis19, Theorem 7.2.16] that, for $x, x' \in \mathrm{F}(A)$, there is a canonical equivalence

$$\operatorname*{colim}_{(x' \to z) \in W_{\mathrm{F}(A),A}(x')} \mathrm{map}_{\mathrm{F}(A)}(x, z) \to \mathrm{map}_{\mathrm{E}(A)}(x, x')$$

and we get an analogous formula for the mapping space in $\mathrm{E}(B)$. So we observe that, in order to prove that $\mathrm{E}(u)$ is fully faithful, it suffices to see that the inclusion of the full subcategory

$$W_{\mathrm{F}(A),A}(x') \hookrightarrow W_{\mathrm{F}(B),B}(\mathrm{F}(u)(x'))$$

is cofinal. To simplify notation, we will consider $\mathrm{F}(A)$ as a full subcategory of $\mathrm{F}(B)$ and omit the $\mathrm{F}(u)$ from the notation. So let $\alpha\colon x' \to z_0$ be in $W_{\mathrm{F}(B),B}(x')$. We would like to show that the comma object

$$W_{\mathrm{F}(A),A}(x')_{\alpha/} = W_{\mathrm{F}(A),A}(x') \times_{W_{\mathrm{F}(B),B}(x')} W_{\mathrm{F}(B),B}(x')_{\alpha/}$$

is weakly contractible.

For this, we make the following observation: Let $\gamma\colon x' \to z$ be any morphism in $W_{\mathrm{F}(B),B}(x')$. Then there is a morphism $\beta\colon x' \to z'$ in $W_{\mathrm{F}(A),A}(x')$ and a morphism $\delta\colon z \to z'$ such that we have $\beta \simeq \delta \circ \gamma$. Let us prove this observation:

Consider the fiber sequence

$$\Omega(z/x') \to x' \xrightarrow{\gamma} z$$

in $\mathrm{F}(B)$. By assumption $\Omega(z/x')$ is in $B \subseteq \mathrm{F}(B)$. By construction of $\mathrm{F}(A)$, we can find a filtered diagram $c_\bullet\colon I \to A$ such that

$$x' \simeq \operatorname*{colim}_I c_i.$$

Because $\Omega(z/x')$ is compact, there is an $i_0 \in I$ such that the map $\Omega(z/x') \to x'$ factors through $c_{i_0} \to x'$. We now consider the cofiber sequence

$$c_{i_0} \to x' \xrightarrow{\beta} x'/c_{i_0},$$

which shows that the map $\beta\colon x' \to x'/c_{i_0}$ is in $W_{\mathrm{F}(A),A}(x')$. But since $\Omega(z/x') \to x'$ factors through $c_{i_0} \to x'$, there is a commutative diagram

$$\begin{array}{ccc} \Omega(z/x') & \longrightarrow & x' \\ \downarrow & & \| \\ c_{i_0} & \longrightarrow & x' \end{array}$$

and so we get an induced morphism of cofibers $\delta\colon z \to x'/c_{i_0}$ with $\beta \simeq \delta \circ \gamma$, as desired.

We will now use this to show that the category $W_{\mathtt{F}(A),A}(x')_{\alpha/}$ is cofiltered an thus contractible. So let $d\colon K \to W_{\mathtt{F}(A),A}(x')_{\alpha/}$ be a finite diagram. Then the composition

$$p\colon K \to W_{\mathtt{F}(A),A}(x')_{\alpha/} \to W_{\mathtt{F}(B),B}(x')_{\alpha/}$$

can be extended to a functor

$$\bar{p}\colon K * \Delta^0 \to W_{\mathtt{F}(B),B}(x')_{\alpha/},$$

as $W_{\mathtt{F}(B),B}(x')$ and thus also $W_{\mathtt{F}(B),B}(x')_{\alpha/}$ have finite colimits. But now the observation shows that we can extend $\bar{p}$ to a diagram

$$p'\colon (K * \Delta^0) \amalg_{\Delta^0} \Delta^1 \to W_{\mathtt{F}(B),B}(x')_{\alpha/}$$

such that the composite

$$\Delta^{\{1\}} \to \Delta^1 \hookrightarrow (K * \Delta^0) \amalg_{\Delta^0} \Delta^1 \xrightarrow{p'} W_{\mathtt{F}(B),B}(x')_{\alpha/}$$

lies in $W_{\mathtt{F}(A),A}(x')_{\alpha/}$. Furthermore, the inclusion

$$(K * \Delta^0) \amalg_{\Delta^0} \Delta^1 \to K * \Delta^1$$

is inner anodyne and we get an induced morphism

$$\tilde{p}\colon K * \Delta^1 \to W_{\mathtt{F}(B),B}(x')_{\alpha/},$$

whose restriction to $K * \Delta^{\{1\}}$ factors through $W_{\mathtt{F}(A),A}(x')_{\alpha/}$ and extends d. Thus we get the claim. □

Proposition 3.47. *Let $A \in \mathrm{Cat}_\infty^{\mathrm{perf}}$. Then the space $K(\mathtt{E}(A))$ is contractible.*

Proof: We note that $\mathtt{E}(A)$ admits countable coproducts. Consider the exact functor G given by the composition

$$\mathtt{E}(A) \xrightarrow{\Delta} \prod_{n\in\mathbb{N}} \mathtt{E}(A) \xrightarrow{\text{coprod}} \mathtt{E}(A).$$

We have a canonical natural transformation $\mathrm{id}_{\mathtt{E}(A)} \to G$, given by the inclusion into the first factor. Then the sequence of functors

$$\mathrm{id}_{\mathtt{E}(A)} \to G \xrightarrow{\mathrm{id}} G$$

is a cofiber sequence. Thus we have that $|S_\bullet(\mathrm{id}_{\mathtt{E}(A)})| + |S_\bullet(G)| \simeq |S_\bullet(G)|$ by Theorem 3.27. Now, adding $|S_\bullet(\Sigma G)|$ shows that $|S_\bullet(\mathrm{id}_{\mathtt{E}(A)})| \simeq 0$ by Example 3.30. □

Construction 3.48. Consider the functors

$$K^n := K \circ \mathtt{E}^n\colon\ \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{S}$$

for all $n \in \mathbb{N}$. By Theorem 3.43 and Proposition 3.47, for any $A \in \mathrm{Cat}_\infty^{\mathrm{perf}}$, the exact sequence

$$A \to \mathtt{F}(A) \to \mathtt{E}(A)$$

gives us a natural equivalence of pointed spaces

$$K(A) \simeq \Omega K(\mathtt{E}(A)).$$

It follows by induction that we get an equivalence of functors

$$\Omega^n \circ K \circ \mathtt{E}^n \simeq K$$

for all n. Thus we get an induced functor

$$\mathbb{K}\colon\ \mathrm{Cat}_\infty^{\mathrm{perf}} \to \lim(\ldots \to S_* \xrightarrow{\Omega} S_* \xrightarrow{\Omega} S_*) \simeq \mathrm{Sp}$$

into the inverse limit, which is equivalent to the ∞-category of spectra. Furthermore, since $\mathtt{E}$ preserves exact sequences and since Ω preserves fiber sequences, the functor $\mathbb{K}$ sends exact sequences to levelwise fiber sequences in the inverse limit and thus to fiber sequences in the category of spectra. Hence the functor $\mathbb{K}$ is localizing. Furthermore, we have that $\Omega^\infty \circ \mathbb{K} = K$ by construction. We call $\mathbb{K}$ the *non-connective K-theory functor*.

4. Algebraic K-Theory of Schemes

This chapter is devoted to studying the algebraic K-theory of schemes. In section 4.1 we will construct a suitable notion of the derived category of quasi-coherent modules $\mathbf{D}(X)$ over a scheme X. We will do this in a way ensuring that the assignment $X \mapsto \mathbf{D}(X)$ is a Zariski sheaf and use this to deduce some important results such as the flat base change formula (Theorem 4.30).

In section 4.2 we will define the subcategory $\mathrm{Perf}(X) \subseteq \mathbf{D}(X)$ of perfect complexes and define the K-Theory of a scheme X to be the K-theory of $\mathrm{Perf}(X)$. We will also show that $\mathbf{D}(X)$ is compactly generated (see Theorem 4.45), which will allow us to apply the results of the third chapter to deduce a localization sequence of non-connective K-theory spectra (see Corollary 4.46).

In section 4.3 we will use these results to conclude that algebraic K-theory satisfies Nisnevich descent.

The key ideas of this chapter are taken from a series of lectures at the University of Regensburg given by Adeel Khan in 2017 (see [Kha17]).

4.1. Quasi-Coherent Modules and Derived Categories

Construction 4.1. Let R be a ring. Consider the category Mod_R whose objects are given by pairs (f, M) where $f\colon S \to R$ is a ring homomorphism and M is an S-module. A morphism of two pairs

$$(f\colon S \to R, M) \to (f'\colon S' \to R, M')$$

is given by a morphism of R-modules

$$M \otimes_S R \to M' \otimes_{S'} R$$

and composition is defined in the obvious way. For a ring homomorphism $\alpha\colon R \to T$, we get an induced functor

$$\alpha^*\colon \mathrm{Mod}_R \to \mathrm{Mod}_T$$

given by mapping an object (f, M) to the object $(\alpha \circ f, M)$ and sending a morphism

$$s\colon (f\colon S \to R, M) \to (f'\colon S' \to R, M')$$

given by $\varphi\colon M \otimes_S R \to M' \otimes_{S'} R$ to the morphism $\alpha^*(s)\colon (\alpha \circ f, M) \to (\alpha \circ f', M')$, which is given by the unique morphism $\hat{\varphi}$ that makes the diagram

$$\begin{array}{ccc} (M \otimes_S R) \otimes_R T & \xrightarrow{\varphi \otimes_R T} & (M' \otimes_{S'} R) \otimes_R T \\ \downarrow{\cong} & & \downarrow{\cong} \\ M \otimes_S T & \xrightarrow{\hat{\varphi}} & M' \otimes_{S'} T \end{array}$$

commute. For a second ring homomorphism $\beta\colon T \to T'$, one checks that we get an equality of functors $\beta^* \circ \alpha^* = (\beta \circ \alpha)^*$. Thus the construction $\mathrm{Mod}_{(-)}$ defines a functor of 1-categories

$$\mathrm{Mod}_{(-)}\colon \mathrm{Ring} \to \mathrm{Cat}\,.$$

Remark 4.2. Let R-Mod denote the ordinary category of R-modules and observe that there is a canonical functor

$$R\text{-Mod} \to \mathrm{Mod}_R$$

given by sending an R-module M to the pair (id_R, M). Note that it has a quasi-inverse

$$\mathrm{Mod}_R \to R\text{-Mod}$$

which sends a pair $(f\colon S \to R, M)$ to $M \otimes_S R$. Thus R-Mod is canonically equivalent to Mod_R and, for a ring homomorphism $\alpha\colon R \to T$, the functor α^* from above is identified with the functor

$$R\text{-Mod} \to T\text{-Mod}$$

given by extension of scalars along α. However, the construction $(-)\text{-Mod}\colon \mathrm{Ring} \to \mathrm{Cat}$ does not give rise to a functor of 1-categories on the nose, as base change is only well defined up to unique isomorphism. The Mod_R-construction is a rectification which fixes this issue.

Let us recall the following well-known result:

Proposition 4.3. *Let R be a ring. Then there is a proper combinatorial model structure on the category of chain complexes of R-modules* $\mathrm{Ch}(R)$*, where*

i) a map is a fibration if and only if it is a level-wise surjection and

ii) a map is a weak equivalence if and only if it is as quasi-isomorphism.

Proof: See [Lur12, Proposition 7.1.2.8]. Here, everything is shown except for right properness, which is just an easy diagram chase. □

Construction 4.4. For a ring homomorphism $f\colon R \to T$, we get an adjunction

$$- \otimes_R S : \mathrm{Ch}(R) \rightleftarrows \mathrm{Ch}(T) : \mathrm{res}^R_T,$$

where the right adjoint preserves fibrations and trivial fibrations. So this is in fact a Quillen adjunction and it follows that the left adjoint preserves cofibrations and weak equivalences between cofibrant objects by Ken Brown's Lemma.

Thus the canonical equivalence $\mathrm{Mod}_R \to R$-Mod equips the category $\mathrm{Ch}(\mathrm{Mod}_R)$ with a model structure such that, for a morphism $f\colon R \to T$, the induced functor

$$\mathrm{Ch}(\mathrm{Mod}_R) \to \mathrm{Ch}(\mathrm{Mod}_T)$$

preserves cofibrations and weak equivalences between cofibrant objects. By $\mathrm{Ch}(\mathrm{Mod}_R)^\circ$ we will denote the full subcategory of $\mathrm{Ch}(\mathrm{Mod}_R)$ spanned by the cofibrant objects. We will write W_R° for the class of weak equivalences between cofibrant objects. By the above, we can now define a 1-functor

$$\Phi\colon \mathrm{Ring} \to \mathrm{sSet}^+$$

to the 1-category of marked simplicial sets by mapping a ring R to the marked simplicial set

$$(N(\mathrm{Ch}(\mathrm{Mod}_R)^\circ)), W_R^\circ),$$

where $N(-)$ denotes the nerve.

4.5. There is a canonical functor of 1-categories

$$i\colon \mathrm{Cat}^1_\infty \to \mathrm{sSet}^+$$

which takes an ∞-category C to the the marked simplicial set $C^\sharp$ whose underlying simplicial set is C and the marked edges are the equivalences. Let us write `Joy` for the class of Joyal equivalences in Cat^1_∞ and `Mark` for the class of marked equivalences in sSet^+. Then i takes Joyal equivalences to marked equivalences and thus induces a functor

$$F\colon \mathrm{Cat}_\infty = \mathrm{Cat}^1_\infty[\mathtt{Joy}^{-1}] \to \mathrm{sSet}^+[\mathtt{Mark}^{-1}].$$

It turns out that F is an equivalence of ∞-categories (see [Lur09, Theorem 3.1.5.1]), so let G denote an inverse. Then one checks that, for $\alpha\colon (C, W) \to (C', W')$ a morphism of marked ∞-categories, the functor $G(\alpha)$ is equivalent to the functor

$$C[W^{-1}] \to C'[W'^{-1}]$$

induced by the universal property of the localization.

Definition 4.6. Let Aff denote the category of affine schemes. We define $\mathbf{D}(-)\colon \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ to be the functor given by the composite

$$\mathrm{Aff}^{\mathrm{op}} = \mathrm{Ring} \xrightarrow{\Phi} \mathrm{sSet}^+ \to \mathrm{sSet}^+[\mathtt{Mark}^{-1}] \xrightarrow{G} \mathrm{Cat}_\infty,$$

where Φ is the functor from Construction 4.4 and G is the functor from 4.5. For a ring R, we call the ∞-category $\mathbf{D}(R)$ the *derived category* of R. For a morphism $f\colon \operatorname{Spec}(A) \to \operatorname{Spec}(B)$ of affine schemes we denote the induced functor $\mathbf{D}(F)\colon \mathbf{D}(B) \to \mathbf{D}(A)$ by f^*.

Remark 4.7. Let us describe the above construction more explicitly: The ∞-category $\mathbf{D}(R)$ is equivalent to the localization

$$\mathrm{Ch}(\mathrm{Mod}_R)^\circ[(W_R^\circ)^{-1}].$$

It follows from [Cis19, Theorem 7.5.18] that $\mathbf{D}(R)$ is equivalent to the ∞-category

$$\mathrm{Ch}(R)[W_R^{-1}]$$

with W_R denoting the collection of all quasi-isomorphisms. Furthermore, we may, for a morphism $f\colon \operatorname{Spec}(T) \to \operatorname{Spec}(R)$ of affine schemes, identify the induced functor

$$f^*\colon \mathbf{D}(R) \to \mathbf{D}(T)$$

with the left derived functor (in the sense of [Cis19, §7.5.23]) of the extension of scalars functor

$$- \otimes_R T\colon \mathrm{Ch}(R) \to \mathrm{Ch}(T).$$

We make the following immediate observations:

Proposition 4.8. *The ∞-category $\mathbf{D}(R)$ is presentable.*

Proof: Since the above model structure on $\mathrm{Ch}(R)$ is combinatorial, this follows from [Cis19, Theorem 7.11.16]. □

Proposition 4.9. *Let $f\colon \operatorname{Spec}(T) \to \operatorname{Spec}(R)$ be a ring homomorphism. Then the induced functor*

$$f^*\colon \mathbf{D}(R) \to \mathbf{D}(T)$$

has a left adjoint.

Proof: This directly follows from Construction 4.4 by [Cis19, Theorem 7.5.30]. □

Combining these two results we get:

Corollary 4.10. *The functor $\mathbf{D}\colon \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ factors through $\mathcal{P}r^L$.*

We furthermore note the following important property:

Proposition 4.11. *The* ∞*-category* $\mathbf{D}(R)$ *is stable.*

Proof: We have already seen that $\mathbf{D}(R)$ has all limits and colimits and it is an easy consequence of [Cis19, Theorem 7.5.18] and its dual that $\mathbf{D}(R)$ is pointed. Thus by [Lur12, Corollary 1.4.2.27], it suffices to show that the loop space functor

$$\Omega\colon \mathbf{D}(R) \to \mathbf{D}(R)$$

is an equivalence. For this, we consider, for any $C_\bullet \in \mathrm{Ch}(R)$, the mapping cone complex $\mathrm{Cone}(C_\bullet)$. We get a canonical pullback square

$$\begin{array}{ccc} C_\bullet[-1] & \longrightarrow & 0 \\ \downarrow{\scriptstyle j} & & \downarrow \\ \mathrm{Cone}(C_\bullet)[-1] & \xrightarrow{p} & C_\bullet \end{array}$$

where p is an epimorphism and all objects are fibrant. Thus the above square is in fact a homotopy pullback square. Furthermore, the object $\mathrm{Cone}(C_\bullet)$ is quasi-isomorphic to the zero-object. It follows that there is a canonical equivalence $C_\bullet[-1] \xrightarrow{\simeq} \Omega C_\bullet$. Thus the shift functor

$$[-1]\colon \mathbf{D}(R) \to \mathbf{D}(R)$$

is equivalent to the loop space functor. Since the shift functor clearly has an inverse by shifting in the other direction, the claim follows. □

The next goal will be to show that the functor $\mathbf{D}(-)$ is a Zariski sheaf of ∞-categories.

Lemma 4.12. *Let* $f_i\colon \mathrm{Spec}\,(B_i) \to \mathrm{Spec}\,(A)$ *be a jointly surjective finite collection of flat morphisms. Then the family* $\{f_i^*\}_i$ *is jointly conservative.*

Proof: Since $\mathbf{D}(\prod_i B_i) \simeq \prod_i \mathbf{D}(B_i)$, we may assume that our collection consists of a single faithfully flat morphism $f\colon \mathrm{Spec}\,(A) \to \mathrm{Spec}\,(B)$. Let $\alpha\colon x \to y$ be a morphism in $\mathbf{D}(A)$. Note that, since f is faithfully flat, the extension of scalars functor

$$- \otimes_B A\colon \mathrm{Ch}(A) \to \mathrm{Ch}(B)$$

preserves quasi-isomorphisms and thus the left derived functor f^* is just the functor induced by the universal property of the localization. So, if the morphism $f^*(\alpha)$ is an equivalence, we have that $\mathrm{cofib}(\alpha) \otimes_B A \simeq 0$ and, in particular,

$$0 \cong H_n(\mathrm{cofib}(\alpha) \otimes_B A) \cong H_n(\mathrm{cofib}(\alpha)) \otimes_B A.$$

Now, since f is faithfully flat, it follows that $H_n(\mathrm{cofib}(\alpha)) \cong 0$ and thus $\mathrm{cofib}(\alpha) \simeq 0$ and the claim follows. □

We are now ready to prove the following descent result:

Proposition 4.13. *Consider a Zariski square*

$$\begin{array}{ccc} \mathrm{Spec}\,(B_3) & \xrightarrow{i'} & \mathrm{Spec}\,(B_1) \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} \\ \mathrm{Spec}\,(B_2) & \xrightarrow{i} & \mathrm{Spec}\,(A) \end{array}$$

of affine schemes (i.e. the square is cartesian, all morphisms are open immersions and $\mathrm{Spec}\,(A) = \mathrm{Spec}\,(B_2) \cup \mathrm{Spec}\,(B_1)$*). Then the induced square*

$$\begin{array}{ccc} \mathbf{D}(A) & \xrightarrow{j^*} & \mathbf{D}(B_1) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i'^*} \\ \mathbf{D}(B_2) & \xrightarrow{j'^*} & \mathbf{D}(B_3) \end{array}$$

is a pullback square of ∞*-categories.*

Proof: An object K in the pullback $\mathbf{D}(B_2) \times_{\mathbf{D}(B_3)} \mathbf{D}(B_1)$ is given by a tuple (K_2, K_1, γ), where $K_i \in \mathbf{D}(B_i)$ and γ is an equivalence

$$i'^*(K_1) \xrightarrow{\gamma} j'^*(K_2)$$

in $\mathbf{D}(B_3)$. We want to show that the induced functor

$$F\colon \mathbf{D}(A) \to \mathbf{D}(B_2) \times_{\mathbf{D}(B_3)} \mathbf{D}(B_1)$$
$$K \mapsto (i^*K, j^*K, \varepsilon_K)$$

is an equivalence. Here, the morphism ε is the natural equivalence witnessing the commutativity of the above diagram. We observe that F has a right adjoint G, where, for an object (K_2, K_1, γ), the image $G(K_2, K_1, \gamma)$ is defined by the pullback square

$$\begin{array}{ccc} G(K_2, K_1, \gamma) & \longrightarrow & j_*K_1 \\ \downarrow & & \downarrow \\ i_*K_2 & \longrightarrow & i_*j'_*j'^*K_2 \end{array}$$

Here, the bottom horizontal map is induced by the unit map $\mathrm{id} \to j'_*j'^*$ and the right vertical map is given by the composite

$$j_*K_1 \to j_*i'_*i'^*K_1 \xrightarrow{j_*i'_*\gamma} j_*i'_*j'^*K_2 \xrightarrow{\simeq} i_*j'_*j'^*K_2.$$

We would now like to see that, for $K \in \mathbf{D}(A)$, the unit map

$$\eta\colon K \to i_*i^*K \times_{i_*j'_*j'^*i^*K} j_*j^*K$$

is an equivalence. By Lemma 4.12, it suffices to see that $i^*\eta$ and $j^*\eta$ are equivalences. Unwinding this, we have to show that

$$K \otimes_A B_1 \to (i_*i^*K \times_{i_*j'_*j'^*i^*K} j_*j^*K) \otimes_A B_1$$

is an equivalence and the same for tensoring with B_2. We compute

$$\begin{aligned}(i_*i^*K \times_{i_*j'_*j'^*i^*K} j_*j^*K) \otimes_A B_1 &\simeq K \otimes_A B_2 \otimes_A B_1 \times_{K\otimes_A B_3 \otimes_A B_1} K \otimes_A B_1 \otimes_A B_1 \\ &\simeq K \otimes_A B_3 \times_{K\otimes_A B_3} K \otimes_A B_1 \\ &\simeq K \otimes_A B_1\end{aligned}$$

and inverse of this equivalence is precisely the map $i^*\eta$. This works similarly if we tensor with B_2 and it follows that F is fully faithful. To prove that F is an equivalence, it now suffices to see that G is conservative. For this, we observe that, for a morphism

$$\varphi\colon (K_2, K_1, \gamma) \to (L_2, L_1, \delta)$$

in $\mathbf{D}(B_2) \times_{\mathbf{D}(B_3)} \mathbf{D}(B_1)$, the morphism $i^*(G(\varphi))$ is equivalent to the morphism given by applying the first projection functor

$$\mathrm{pr}_1\colon \mathbf{D}(B_2) \times_{\mathbf{D}(B_3)} \mathbf{D}(B_1) \to \mathbf{D}(B_2)$$

to φ. The analogue holds for $j^*(G(\varphi))$ and the second projection. It follows that G is conservative, since the functor

$$\mathrm{pr}_1 \times \mathrm{pr}_2\colon \mathbf{D}(B_2) \times_{\mathbf{D}(B_3)} \mathbf{D}(B_1) \to \mathbf{D}(B_2) \times \mathbf{D}(B_1)$$

is conservative. □

From now, on let us fix a quasi-compact and quasi-separated scheme S.

Definition 4.14. Let $\mathrm{Aff}_{/S}$ denote the category of finitely presented affine S-schemes. We equip $\mathrm{Aff}_{/S}$ with a topology as follows: A finite collection $\{f_i\colon U_i \to X\}_i$ generates a covering sieve, if

- every f_i is an open immersion and
- the induced morphism $\coprod_i U_i \to X$ is surjective.

We call the resulting topology the *affine Zariski topology* and $\mathrm{Aff}_{/S}$ equipped with this topology the *(big) affine Zariski site*.

One has an analogous version of Theorem 1.15 for the Zariski topology. The proof is similar as well and will be skipped here.

Theorem 4.15. *Let $\mathcal{F} \in \mathrm{Psh}(\mathrm{Aff}_{/S})$. Then $\mathcal{F}$ is a sheaf with respect to the affine Zariski topology if and only if, for every Zariski square of affine schemes*

$$\begin{array}{ccc}\mathrm{Spec}\,(B_3) & \xrightarrow{j'} & \mathrm{Spec}\,(B_1) \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\ \mathrm{Spec}\,(B_2) & \xrightarrow{j} & \mathrm{Spec}\,(A)\end{array}$$

the induced square

$$\begin{array}{ccc} \mathcal{F}(\operatorname{Spec}(A)) & \longrightarrow & \mathcal{F}(\operatorname{Spec}(B_1)) \\ \downarrow & & \downarrow \\ \mathcal{F}(\operatorname{Spec}(B_2)) & \longrightarrow & \mathcal{F}(\operatorname{Spec}(B_3)) \end{array}$$

is a pullback square in $\mathcal{S}$.

4.16. We will now briefly discuss sheaves of ∞-categories: Let (C, τ) be a site with pullbacks. We will call a functor

$$F : C \to \mathrm{Cat}_\infty$$

a τ-sheaf of ∞-categories if, for every covering family $\{U_i \to X\}_i$, the canonical map

$$F(X) \to \lim_{n \in \mathbf{\Delta}} \prod_{(i_1,\dots,i_n)} F(U_{i_1} \times_X \ldots \times_X U_{i_n})$$

is an equivalence. We write $\mathrm{Sh}_\tau(C, \mathrm{Cat}_\infty) \subseteq \mathrm{Fun}(C, \mathrm{Cat}_\infty)$ for the full subcategory spanned by the τ-sheaves of ∞-categories.

Recall that to an ∞-category C one can associate a simplicial space

$$\begin{aligned} N_{\mathrm{segal}}(C) \colon \mathbf{\Delta}^{\mathrm{op}} &\to \mathcal{S} \\ [n] &\mapsto \mathrm{Fun}(\Delta^n, C)^{\simeq}. \end{aligned}$$

It is a well-known fact that this defines a fully faithful functor

$$N_{\mathrm{segal}} \colon \mathrm{Cat}_\infty \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S}),$$

whose essential image is spanned by the so called *complete segal spaces*. These are all the functors $F : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}$ such that, for all $m \in \mathbb{N}_{>0}$, the induced map

$$F([m]) \to F(\{0 < 1\}) \times_{F(1)} \ldots \times_{F(n-1)} F(\{n-1 < n\})$$

is an equivalence. Furthermore, if we denote by J the unique ∞-groupoid with two objects and one equivalence between them, then the map

$$D_0 \to \mathrm{map}_{\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})}(N_{\mathrm{segal}}(J), F)$$

induced by $I \to \{0\}$ is an equivalence. For the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})$ spanned by the complete segal spaces, we will write **CSS**. Furthermore, we observe that the inclusion

$$\mathbf{CSS} \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})$$

preserves all limits and therefore limits of complete segal spaces are computed pointwise. It follows that we may identify the full subcategory

$$\mathrm{Sh}_\tau(C, \mathrm{Cat}_\infty) \subseteq \mathrm{Fun}(C, \mathrm{Cat}_\infty)$$

with the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Sh}_\tau(C))$ spanned by all functors $F\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Sh}_\tau(C)$ such that, for all $c \in C$, the functor

$$F(-)(c)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}$$

is a complete segal space. This description allows us to upgrade many results, such as Theorem 4.15 and Proposition 4.21 below, from sheaves of spaces to sheaves of ∞-categories.

Corollary 4.17. *The functor* $\mathbf{D}\colon\ \mathrm{Aff}_{/S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ *is a Zariski sheaf of ∞-categories on* $\mathrm{Aff}_{/S}$.

Our next goal is to extend the construction $\mathbf{D}(-)$ to non-affine schemes.

Definition 4.18. Let $\mathrm{Sch}_{/S}$ denote the category of all finitely presented S-schemes. We equip $\mathrm{Sch}_{/S}$ with a topology as follows: A finite collection $\{f_i\colon U_i \to X\}_i$ generates a covering family if

- every f_i is an open immersion and
- the induced morphism $\coprod_i U_i \to X$ is surjective.

We call the resulting topology the *Zariski topology*, and $\mathrm{Sch}_{/S}$ equipped with this topology the *(big) Zariski site*.

Remark 4.19. The obvious analogue of Theorem 4.15 still holds in the non-affine case.

4.20. The inclusion $i\colon\ \mathrm{Aff}_{/S} \hookrightarrow \mathrm{Sch}_{/S}$ preserves pullbacks. Also it induces an adjunction of ∞-categories of presheaves

$$c^i\colon\ \mathrm{Psh}(\mathrm{Sch}_{/S}) \rightleftarrows \mathrm{Psh}(\mathrm{Aff}_{/S})\ : i_*,$$

where c^i denotes the functor given by precomposition with i and where i_* is given by right Kan-extension. Furthermore, any object in $\mathrm{Sch}_{/S}$ admits a covering by objects in $\mathrm{Aff}_{/S}$, thus [Hoy15, Lemma C.3] implies:

Proposition 4.21. *The adjunction* $c^i \dashv i_*$ *restricts to an equivalence of ∞-categories*

$$\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sch}_{/S}) \xrightarrow{\simeq} \mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Aff}_{/S}).$$

In particular, a presheaf $\mathcal{F}$ *on* $\mathrm{Sch}_{/S}$ *is a Zariski sheaf if and only if it is the right Kan-extension of a Zariski sheaf on* $\mathrm{Aff}_{/S}$.

Corollary 4.22. *A functor* $F\colon\ \mathrm{Sch}_{/S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ *is a Zarsiki sheaf of ∞-categories if and only if it is the right Kan-extension of a Zariski sheaf on* $\mathrm{Aff}_{/S}$.

Definition 4.23. We define the functor $\mathbf{D}\colon\ \mathrm{Sch}_{/S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ to be the right Kan-extension along $i\colon\ \mathrm{Aff}_{/S}^{\mathrm{op}} \to \mathrm{Sch}_{/S}^{\mathrm{op}}$ of the functor $\mathbf{D}\colon\ \mathrm{Aff}_{/S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ from Definition 4.6. We call $\mathbf{D}(X)$ the *derived category* of X.

Remark 4.24. It follows from Corollary 4.22 that **D** is a Zariski sheaf on $\mathrm{Sch}_{/S}$. Spelling out the definition, we see that, when restricted to affine schemes, the functor **D** is equivalent to $\mathbf{D}\colon \mathrm{Aff}_{/S} \to \mathrm{Cat}_\infty$ as defined before. For a general scheme $X \in \mathrm{Sch}_{/S}$, the canonical map

$$\mathbf{D}(X) \to \lim_{\mathrm{Spec}(A)\in\mathrm{Aff}_{/X}} \mathbf{D}(A)$$

is an equivalence.

Remark 4.25. We will now compare our derived ∞-category $\mathbf{D}(X)$ with an alternative, more classical definition. For this, consider the abelian category Mod_X of all $\mathcal{O}_X$-modules. This is a Grothendieck abelian category and it follows that the localization of the category $\mathrm{Ch}(\mathrm{Mod}_X)$ at the quasi-isomorphisms is a presentable stable ∞-category $\mathbf{D}(\mathrm{Mod}_{\mathcal{O}_X})$ (see [Lur12, Proposition 1.3.5.15] and [Lur12, Proposition 1.3.5.21]). We then define $\mathbf{D}'(X)$ to be the full subcategory of $\mathbf{D}(\mathrm{Mod}_{\mathcal{O}_X})$ spanned by all complexes whose homology objects are quasi-coherent $\mathcal{O}_X$-modules. In order to see that $\mathbf{D}'(X)$ and $\mathbf{D}(X)$ agree, we observe that an argument similar to the one given in Proposition 4.13 shows that the construction

$$\mathbf{D}'\colon \mathrm{Sch}_{/X}^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

defines a Zariski sheaf. Thus, it suffices to see that the two constructions agree in the affine case. This follows as in [Sta20, Tag 06Z0]. In particular, it follows that the homotopy category $\mathrm{ho}(\mathbf{D}(X))$ agrees with the classical derived category $\mathbf{D}_{\mathrm{Qcoh}}(X)$ as defined in [Sta20, Tag 06YZ].

We will now study further properties of the sheaf **D**. We start with the following easy observation:

Proposition 4.26. *The functor* $\mathbf{D}\colon \mathrm{Sch}_{/S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ *factors through* $\mathcal{P}r^L_{\mathrm{st}}$.

Proof: This is clear from Corollary 4.10 since the inclusion

$$\mathcal{P}r^L_{\mathrm{st}} \to \mathrm{Cat}_\infty$$

preserves all limits by [Lur09, Theorem 5.5.3.13] and [Lur12, Theorem 1.1.4.4]. □

Notation 4.27. Let $f\colon X \to Y$ be a morphism in $\mathrm{Sch}_{/S}$. As above, we will denote the induced functor

$$\mathbf{D}(f)\colon \mathbf{D}(Y) \to \mathbf{D}(X)$$

by f^* and call it the *inverse image functor*. By Proposition 4.26, it admits a right adjoint

$$f_*\colon \mathbf{D}(X) \to \mathbf{D}(Y),$$

which we will call the *direct image functor*.

4.28. Let $X \in \mathrm{Sch}_{/S}$ be a scheme and let $X = U \cup V$ be an open covering. Let j_U and j_V denote the inclusions from U and V, respectively, to X. For $K \in \mathbf{D}(X)$, we will write $K|_U$ for $j_U^*(K)$ and $K|_V$ for $j_V^*(K)$. We observe that, since **D** satisfies Zariski descent, the canonical functor

$$F\colon \mathbf{D}(X) \to \mathbf{D}(U) \times_{\mathbf{D}(U\cap V)} \mathbf{D}(V)$$

is an equivalence. We note that, as in the proof of Proposition 4.13, we can write down an explicit right adjoint to F such that the unit is an equivalence. Spelling out the details, we see that the unit is given by the canonical morphism

$$K \to j_{U*}\, K|_U \times_{j_{U\cap V*}\, K|_{U\cap V}} j_{V*}\, K|_V$$

which is therefore invertible. In particular, it follows that, if we are given a morphism $f\colon X \to Y$, then the canonical morphism

$$f_* K \to (f \circ j_U)_*\, K|_U \times_{(f\circ j_{U\cap V})_*\, K|_{U\cap V}} (f \circ j_V)_* K$$

is an equivalence.

Let us quickly recall the following terminology from [Kha16, Ch. 0, §3.3]:

Definition 4.29. We will say that a commutative square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f^*} & \mathcal{D} \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle q^*} \\ \mathcal{C}' & \xrightarrow{p^*} & \mathcal{D}' \end{array}$$

is *vertically right adjointable* if the base change natural transformation

$$f^* g_* \to q_* p^*$$

is invertible.

We are now ready to proof the following important result about the compatibility of the inverse and the direct image functor:

Theorem 4.30 (Flat base change)**.** *Consider a pullback square*

$$\begin{array}{ccc} X' & \xrightarrow{p} & Y' \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

in $\mathrm{Sch}_{/S}$*, where g is flat and f is quasi-compact and quasi-separated. Then the induced square*

$$\begin{array}{ccc} \mathbf{D}(Y) & \xrightarrow{g^*} & \mathbf{D}(Y') \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle p^*} \\ \mathbf{D}(X) & \xrightarrow{q^*} & \mathbf{D}(X'). \end{array}$$

is vertically right adjointable.

Proof: Let us start by assuming that all schemes in discussion are affine, so that we are given a square

$$\begin{array}{ccc} \mathrm{Spec}\,(T) & \xrightarrow{p} & \mathrm{Spec}\,(S) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle g} \\ \mathrm{Spec}\,(A) & \xrightarrow{f} & \mathrm{Spec}\,(B) \end{array}$$

For $K \in \mathbf{D}(A)$, the base change natural transformation is given by the canonical equivalence

$$K \otimes_B S \xrightarrow{\simeq} (K \otimes_A A) \otimes_B S \xrightarrow{\simeq} K \otimes_A (A \otimes_B S) \simeq K \otimes_A T,$$

which proves the affine case.

The strategy of the proof is to reduce the situation to the affine case. We will start by reducing to the case where Y' and Y are affine. For this, we pick an open covering $Y = U_1 \cup \ldots \cup U_k$ by finitely many affine opens. Let us write $U_i' = U_i \times_Y X$. Now, by Zariski descent, the canonical functors

$$\mathbf{D}(Y) \to \lim_{n \in \mathbf{\Delta}} \prod_{(i_1,\ldots,i_n)} \mathbf{D}(U_{i_1} \times_Y \ldots \times_Y U_{i_n})$$

and

$$\mathbf{D}(X) \to \lim_{n \in \mathbf{\Delta}} \prod_{(i_1,\ldots,i_n)} \mathbf{D}(U'_{i_1} \times_X \ldots \times_X U'_{i_n})$$

are equivalences. For every injective morphism $\alpha\colon [n] \to [m]$ in $\mathbf{\Delta}$, we consider the square

$$\begin{array}{ccc} \prod_{(i_1,\ldots,i_n)} \mathbf{D}(U_{i_1} \times_Y \ldots \times_Y U_{i_n}) & \longrightarrow & \prod_{(i_1,\ldots,i_m)} \mathbf{D}(U_{i_1} \times_Y \ldots \times_Y U_{i_m}) \\ \downarrow & & \downarrow \\ \prod_{(i_1,\ldots,i_n)} \mathbf{D}(U'_{i_1} \times_X \ldots \times_X U'_{i_n}) & \longrightarrow & \prod_{(i_1,\ldots,i_m)} \mathbf{D}(U'_{i_1} \times_X \ldots \times_X U'_{i_m}) \end{array} \tag{1}$$

and claim that it is vertically right adjointable. This square is the product of the squares

$$\begin{array}{ccc} \mathbf{D}(U'_{j_1} \times_Y \ldots \times_Y U_{j_n}) & \longrightarrow & \prod_{(i_1,\ldots,i_m) \in I} \mathbf{D}(U_{i_1} \times_Y \ldots \times_Y U_{i_m}) \\ \downarrow & & \downarrow \\ \mathbf{D}(U'_{j_1} \times_X \ldots \times_X U'_{j_n}) & \longrightarrow & \prod_{(i_1,\ldots,i_m) \in I} \mathbf{D}(U'_{i_1} \times_X \ldots \times_X U'_{i_m}) \end{array} \tag{2}$$

for every $(j_1, \ldots, j_n) \in \{1, \ldots, k\}^m$ where I is the set of all $(i_1, \ldots i_m) \in \{1, \ldots, k\}^n$ that agree with $(j_0, \ldots, j_n)$ when restricted along α. Now, since taking right adjoints commutes with products, it suffices to show that (2) is vertically right adjointable. Again, since taking products commutes with taking right adjoints, we may reduce to the squares

$$\begin{array}{ccc}
\mathbf{D}(U_{j_1} \times_Y \ldots \times_Y U_{j_n}) & \longrightarrow & \mathbf{D}(U_{i_1} \times_Y \ldots \times_Y U_{i_m}) \\
\downarrow & & \downarrow \\
\mathbf{D}(U'_{j_1} \times_X \ldots \times_X U'_{j_n}) & \longrightarrow & \mathbf{D}(U'_{i_1} \times_X \ldots \times_X U'_{i_m})
\end{array}$$

for all $(i_1, \ldots, i_m) \in I$, that are induced by the pullback squares of schemes

$$\begin{array}{ccc}
U'_{i_1} \times_X \ldots \times_X U'_{i_m} & \longrightarrow & U_{i_1} \times_Y \ldots \times_Y U_{i_m} \\
\downarrow & & \downarrow \\
U'_{j_1} \times_X \ldots \times_X U'_{j_n} & \longrightarrow & U_{j_1} \times_Y \ldots \times_Y U_{j_n}
\end{array}$$

Here, the vertical arrows are open immersions and the horizontal arrows are quasi-compact and quasi-separated. As Y is not necessarily separated, the intersection $U_{j_1} \times_Y \ldots \times_Y U_{j_n}$ might not be affine. The intersection is, however, separated.

Let us assume that we have already proven the theorem for the special case that the bottom right corner of the given pullback square is separated. Then (1) is vertically right adjointable. It then follows from [Lur12, Corollary 4.7.4.18] and [Lur09, Lemma 6.3.5.7] that the square

$$\begin{array}{ccc}
\mathbf{D}(Y) & \longrightarrow & \prod_i \mathbf{D}(U_i) \\
\downarrow & & \downarrow \\
\mathbf{D}(X) & \longrightarrow & \prod_i \mathbf{D}(U'_i)
\end{array} \tag{3}$$

is vertically right adjointable. So in the cube

$$\begin{array}{ccccccc}
 & & \coprod_i X' \times_X U'_i & & \longrightarrow & & \coprod_i Y' \times_Y U_i \\
 & \swarrow & & & & \swarrow & \\
X' & & \longrightarrow & & Y' & & \downarrow \\
 & & \downarrow & & & & \\
\downarrow & & \coprod_i U'_i & & \longrightarrow & & \coprod_i U_i \\
 & \swarrow & & & \downarrow & \swarrow & \\
X & & \longrightarrow & & Y & &
\end{array}$$

the statement of the theorem holds for the bottom and the back square. By Zariski descent, restriction to an open covering is conservative, so the statement of the theorem holds for the front square if it holds for the top square. But we can refine the covering $\{Y' \times_Y U_i \to Y'\}$ by affine schemes and conclude that the statement of the theorem holds for the top square, because we have seen above that squares like (3) are vertically right adjointable. So the statement of the theorem holds for the front square.

Hence, if we prove the theorem for Y separated, we are done. We can now rerun the argument above with Y separated, but then the intersections $U_{j_1} \times_Y \ldots \times_Y U_{j_n}$ are affine and thus we may reduce to Y being affine. Furthermore, we can refine Y' by an affine open covering and thus reduce to Y' being affine as well. This completes the first reduction step.

Now we can cover X by finitely many affine opens and use induction over the size of the cover. We already know that the theorem holds in the affine case and, by induction, we may assume that there are open immersions $j_U\colon U = \operatorname{Spec}(A) \to X$ and $j_V\colon V = \cup_{i=1}^n \operatorname{Spec}(B_i) \to X$, such that the theorem holds for the outer square of the commutative diagram

$$\begin{array}{ccccc} X'_U & \xrightarrow{j'_U} & X' & \xrightarrow{p} & Y' \\ \downarrow{\scriptstyle q_U} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{j_U} & X & \xrightarrow{f} & Y \end{array}$$

and similarly for V. Now, we observe that in the commutative square

$$\begin{array}{ccc} \mathbf{D}(X_U) \times_{\mathbf{D}(X_{U\cap V})} \mathbf{D}(X_V) & \xleftarrow{({j'_U}^*,{j'_V}^*)} & \mathbf{D}(X') \\ {\scriptstyle q_U^* \times_{q^*_{U\cap V}} q_V^*}\uparrow & & {\scriptstyle q^*}\uparrow \\ \mathbf{D}(U) \times_{\mathbf{D}(U\cap V)} \mathbf{D}(V) & \xleftarrow[(j_U^*,j_V^*)]{} & \mathbf{D}(X) \end{array}$$

the horizontal arrows are equivalences by Zariski descent and thus we have that, for $K \in \mathbf{D}(X)$,

$$q^*K \simeq j'_{U*}(q_U^*\ K|_U) \times_{j'_{U\cap V*}(q^*_{U\cap V}\ K|_{U\cap V})} j'_{V*}(q_V^*\ K|_V),$$

as in 4.28. If we apply p_*, we see that

$$p_*(j'_{U*}(q_U^*\ K|_U)) \simeq g^*((f \circ j_U)_*\ K|_U),$$

by the affine case and similarly for V, by the induction hypothesis. Thus, by 4.28, it remains to see that

$$p_* j'_{U\cap V*}(q^*_{U\cap V}\ K|_{U\cap V}) \simeq g^*((f \circ j_{U\cap V})_*\ K|_{U\cap V}).$$

Note that this is not clear, as we may not be able to cover $U \cap V$ by n affine opens, as the intersections $\operatorname{Spec}(A) \cap \operatorname{Spec}(B_i)$ may not be affine if X is not separated. So we can not apply the induction hypothesis immediately. However, we observe that it suffices to see that the theorem holds for the square

$$\begin{array}{ccc} X'_{U\cap V} & \longrightarrow & X'_U \\ \downarrow{\scriptstyle q_{U\cap V}} & & \downarrow{\scriptstyle q_U} \\ U \cap V & \longrightarrow & U. \end{array}$$

But since U is affine, the intersection $U \cap V$ is separated, so we can now rerun the above argument to finish the proof. □

Corollary 4.31. *Let $j\colon U \to X$ be a quasi-compact open immersion. Then j_* is fully faithful.*

Proof: This follows from applying Theorem 4.30 to the pullback square

$$\begin{array}{ccc} U & = & U \\ \| & & \downarrow \\ U & \hookrightarrow & X \end{array}$$

□

Proposition 4.32. *Let S be a quasi-compact quasi-separated scheme and let $f\colon X \to Y$ be a morphism in $\mathrm{Sch}_{/S}$ (in particular, the morphism f is quasi-compact and quasi-separated). Then $f_*\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ preserves all colimits.*

Proof: Let us first assume that $X = \mathrm{Spec}\,(B)$ and $Y = \mathrm{Spec}\,(A)$ are affine. In this case, the functor

$$f_*\colon \mathbf{D}(B) \to \mathbf{D}(A)$$

is induced by the restriction along the ring homomorphism $A \to B$. Furthermore, we have already seen that f_* is exact, so it suffices to see that f_* preserves coproducts. Since coproducts in $\mathbf{D}(B)$ are computed as ordinary coproducts in $\mathrm{Ch}(B)$, this is clear.

We now reduce the general situation to the affine case: The proof is very similar to the proof of Theorem 4.30, but let us quickly sketch it: First of all, we may immediately reduce to Y being affine by applying Theorem 4.30. Again, we use induction to write $X = U \cup V$, where $U = \mathrm{Spec}\,(A)$ and $\mathrm{Spec}\,(B) = \mathrm{Spec}\,(B_1) \cup \ldots \cup \mathrm{Spec}\,(B_n)$. Then we assume that the claim holds for $f \circ j_U$ and $f \circ j_V$. By 4.28, it suffices to show the claim for the composition

$$U \cap V \hookrightarrow X \xrightarrow{f} Y.$$

But now, the intersection $U \cap V$ is separated. So we just rerun the above argument and get the claim. □

4.2. Perfect Complexes and Compact Generation

Definition 4.33. Let R be a commutative ring. A chain complex $C_\bullet$ of R-modules is called a *perfect complex* if it is quasi-isomorphic to a bounded complex of finitely generated projective R-modules. We will denote by $\mathrm{Perf}(R)$ the full subcategory of $\mathbf{D}(R)$ spanned by the perfect complexes.

Proposition 4.34. *Let R be a ring. Then $\mathrm{Perf}(R) \subseteq \mathbf{D}(R)$ is the full subcategory spanned by the compact objects of $\mathbf{D}(R)$.*

Proof: The classical statement in [Sta20, Tag 07LT] tells us that the objects of $\mathrm{Perf}(R)$ are precisely the compact objects in the homotopy category $\mathrm{ho}(\mathbf{D}(R))$, in the sense that, for any $\mathcal{F} \in \mathrm{Perf}(R)$, the canonical map

$$\bigoplus_{i\in I} \mathrm{Hom}_{\mathrm{ho}(\mathbf{D}(R))}(F, E_i) \to \mathrm{Hom}_{\mathrm{ho}(\mathbf{D}(R))}\left(F, \bigoplus_{i\in I} E_i\right)$$

is bijective. But now [Lur12, Proposition 1.4.4.1] shows that this is equivalent to $\mathcal{F}$ being a compact object in the ∞-category $\mathbf{D}(R)$. □

We will now extend our definitions to schemes.

Definition 4.35. Let X be a quasi-compact quasi-separated scheme. We define $\mathrm{Perf}(X)$ to be the full subcategory of $\mathbf{D}(X)$ consisting of those complexes $\mathcal{F}$ such that there is a covering $\{U_i = \mathrm{Spec}\,(A_i)\}_{i\in I}$ by affine opens with $j_i^*(\mathcal{F}) \in \mathbf{D}(A_i)$ a perfect complex. Here, the map $j_i\colon U_i \to X$ denotes the inclusion.

We now get the following generalization of Proposition 4.34:

Proposition 4.36. *Let X be a quasi-compact quasi-separated scheme. Then* $\mathrm{Perf}(X) \subseteq \mathbf{D}(X)$ *is given by the full subcategory spanned by the compact objects of* $\mathbf{D}(X)$.

Proof: Let us first assume that $\mathcal{F}$ is a compact object of $\mathbf{D}(X)$. Now let $\{U_i\}_{i\in I}$ be any covering of X by affine opens. By Proposition 4.32, the functor $j_i^*\colon \mathbf{D}(X) \to \mathbf{D}(U_i)$ has a right adjoint which preserves colimits and therefore j_i^* preserves compact objects. So by Lemma 4.34, it follows that $j_i^*(\mathcal{F})$ is a perfect complex and thus $\mathcal{F} \in \mathrm{Perf}(X)$.

Let us now prove the converse. By assumption, there is a cover of X by finitely many affine opens $\{\mathrm{Spec}\,(A_i)\}_{i=1,\dots,n}$ such that $j_i^*(\mathcal{F})$ is a perfect complex for all i. We now proceed by induction on n. For $n = 1$, the claim follows from Proposition 4.34. For $n > 1$, we write U for $\mathrm{Spec}\,(A_1)$ and

$$V := \bigcup_{i=2,\dots,n} \mathrm{Spec}\,(A_i)$$

such that $X = U \cup V$. Then, by induction, both $j_V^*(\mathcal{F})$ and $j_U^*(\mathcal{F})$ are compact. By Remark 4.24, we have a pullback square

$$\begin{array}{ccc} \mathbf{D}(X) & \xrightarrow{j_V^*} & \mathbf{D}(V) \\ \downarrow{\scriptstyle j_U^*} & & \downarrow \\ \mathbf{D}(U) & \longrightarrow & \mathbf{D}(U\cap V) \end{array}$$

and thus a natural equivalence of functors

$$\mathrm{map}_{\mathbf{D}(X)}(\mathcal{F},-) \simeq \mathrm{map}_{\mathbf{D}(U)}(j_U^*(\mathcal{F}), j_U^*(-)) \times_{\mathrm{map}_{\mathbf{D}(U\cap V)}(j_U^*(\mathcal{F})|_{U\cap V},\, j_U^*(-)|_{U\cap V})} \mathrm{map}_{\mathbf{D}(V)}(j_V^*(\mathcal{F}), j_V^*(-)).$$

Since restriction to $U \cap V$ preserves compact objects, all components in this fiber product preserve filtered colimits. Thus the claim follows, as filtered colimits commute with finite limits in $\mathcal{S}$. □

Remark 4.37. By Proposition 4.32, it follows that, for a quasi-compact and quasi-separated morphism $f\colon X \to Y$, the inverse image f^* preserves compact objects. Therefore, by the above Proposition, it restricts to a functor

$$\mathrm{Perf}(f)\colon\ \mathrm{Perf}(Y) \to \mathrm{Perf}(X).$$

Furthermore, it follows from the above proposition that, if U and V are quasi-compact open subschemes of X and $X = U \cup V$, then the induced square

$$\begin{array}{ccc} \mathrm{Perf}(X) & \longrightarrow & \mathrm{Perf}(V) \\ \downarrow & & \downarrow \\ \mathrm{Perf}(U) & \longrightarrow & \mathrm{Perf}(U \cap V) \end{array}$$

is a pullback square. Thus the functor

$$\mathrm{Perf}\colon \mathrm{Sch}_{/S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

is a Zariski sheaf.

Definition 4.38. Let X be a quasi-compact and quasi-separated scheme. We define the *algebraic K-theory space of* X to be $K(X) := K(\mathrm{Perf}(X))$. Similarly, we define the *non-connective algebraic K-theory spectrum* of X to be $\mathbb{K}(\mathrm{Perf}(X))$.

Definition 4.39. Let X be a quasi-compact and quasi-separated scheme and let U be a quasi-compact open subscheme. Then we define $\mathbf{D}(X)_U$ to be the full subcategory of all objects $\mathcal{F} \in \mathbf{D}(X)$ with $\mathcal{F}|_U \simeq 0$. Furthermore, we define $\mathrm{Perf}(X)_U = \mathbf{D}(X)_U^{\mathrm{comp}}$ and $K(X)_U = K(\mathrm{Perf}(X)_U)$ and similarly $\mathbb{K}(X)_U = \mathbb{K}(\mathrm{Perf}(X)_U)$.

We would now like to apply the results of the last section to deduce a cofiber sequence

$$\mathbb{K}(X)_U \to \mathbb{K}(X) \to \mathbb{K}(U).$$

Our first goal is to show that $\mathbf{D}(X)_U$ is compactly generated. For this, we will use the following convenient criterion:

Proposition 4.40. *Let C be a presentable stable ∞-category. Then C is compactly generated if and only if there is a set of compact objects $S \subseteq \mathrm{Ob}(C)$ such that an object $F \in C$ is zero if and only if*

$$\pi_0(\mathrm{map}_C(\Sigma^n X, F)) = \{*\}$$

for all $X \in S$ and $n \in \mathbb{Z}$.

Proof: It is clear that, if C is compactly generated, the set of compact objects satisfies the assumption.

For the converse, let us consider the full subcategory C_0 of C that is spanned by finite colimits of objects of the form $\Sigma^n X$ for $X \in S$ and $n \in \mathbb{Z}$. In particular, all objects in C_0 are compact. Thus the induced functor $F\colon \mathrm{Ind}(C_0) \to C$ is fully faithful and preserves colimits. Hence it has a right adjoint G. Let us now pick $A \in C$. We would like to show that the counit

$$\eta\colon FGA \to A$$

is an equivalence. For this, we observe that $G(\mathrm{cofib}(\eta)) \simeq 0$ and, since F is fully faithful, it follows that

$$\mathrm{map}_C(\Sigma^n X, \mathrm{cofib}(\eta)) \simeq *$$

for all $X \in S$ and $n \in \mathbb{Z}$, as $\Sigma^n X \in C_0$. By assumption, it follows that $\mathrm{cofib}(\eta) \simeq 0$ and so η is an equivalence as C is stable. □

Definition 4.41. We will say that a presentable stable ∞-category C is *compactly generated by a single object* if, in the above proposition, the set S can be chosen to have one element. We call such an object a *compact generator* of C.

Proposition 4.42. *Let R be a commutative ring. Then* $\mathbf{D}(R)$ *is compactly generated by a single object.*

Proof: We know that $\mathbf{D}(R)$ is presentable. We claim that the complex consisting only of R concentrated in degree zero is a compact generator of $\mathbf{D}(R)$. Compactness is clear as R is a projective R-module. Note that, for any $X \in \mathbf{D}(R)$, we have that

$$\pi_0(\mathrm{map}_{\mathbf{D}(R)}(\Sigma^n R, X) \cong \mathrm{Hom}_{\mathrm{ho}(\mathbf{D}(R))}(\Sigma^n R, X) \cong H_n(X).$$

If $H_n(X)$ is 0 for all $n \in \mathbb{Z}$, we get that $X \simeq 0$. □

Construction 4.43. Consider the 1-category $\mathrm{Ch}(R)$ of complexes of R-modules. Let P be a bounded below complex of projective modules. Then the functor

$$- \otimes_R P\colon\ \mathrm{Ch}(R) \to \mathrm{Ch}(R)$$

is exact and thus preserves quasi-isomorphisms. Furthermore, it has a right adjoint

$$\underline{\mathrm{Hom}}_R(P, -)\colon\ \mathrm{Ch}(R) \to \mathrm{Ch}(R)$$

and this induces an adjunction on derived categories

$$- \otimes_R P\colon \mathbf{D}(R) \rightleftarrows \mathbf{D}(R) : \underline{\mathrm{Hom}}(P, -).$$

Proposition 4.44. *Let* $X = \mathrm{Spec}\,(R)$ *be an affine scheme and let* $j\colon U \hookrightarrow X$ *be a quasi-compact open subscheme. Then* $\mathbf{D}(X)_U$ *is compactly generated by a single perfect complex.*

Proof: Since U is quasi-compact, there are $f_1, \ldots, f_n \in R$ such that

$$X = \mathrm{Spec}\,(R_1) \cup \ldots \cup \mathrm{Spec}\,(R_n)\,,$$

where we write $R_i = R[\frac{1}{f_i}]$. Consider for all $1 \leq i \leq n$ the cofiber of the map

$$R \xrightarrow{\cdot f_i} R,$$

regarded as a map of chain complexes concentrated in degree zero. Then $\mathrm{cofib}(\cdot f_i)$ is a finite colimit of compact objects and thus itself compact. Therefore it is equivalent to a bounded complex of finitely generated projective R-modules, say K_i. Now we set

$$\mathcal{K} \coloneqq K_1 \otimes_R \ldots \otimes_R K_n$$

and claim that this is a compact generator. First of all, we observe that $\mathcal{K}$ is in fact in $\mathbf{D}(X)_U$, since $j^*(\mathcal{K})|_{\mathrm{Spec}(R_i)} \simeq \mathcal{K} \otimes_R R[\frac{1}{f_i}]$ is zero for all i, because $K_i \otimes_R R_i \simeq 0$. Now let $\mathcal{F} \in \mathbf{D}(X)_U$ such that

$$\pi_0(\mathrm{map}_{\mathbf{D}(X)}(\Sigma^n \mathcal{K}, \mathcal{F})) \simeq 0$$

for all $n \in \mathbb{Z}$. We want to conclude that $\mathcal{F} \simeq 0$. By the above discussion, we see that

$$0 = \pi_0(\mathrm{map}_{\mathbf{D}(X)}(\Sigma^n \mathcal{K}, \mathcal{F})) = \pi_0(\mathrm{map}_{\mathbf{D}(X)}(\Sigma^n K_1, \underline{\mathrm{Hom}}(K_2 \otimes_R \ldots \otimes_R K_n, \mathcal{F}))).$$

But, by construction of K_1, this implies that the map

$$\cdot f_1 \colon H_n(\underline{\mathrm{Hom}}(K_2 \otimes_R \ldots \otimes_R K_n, \mathcal{F})) \to H_n(\underline{\mathrm{Hom}}(K_2 \otimes_R \ldots \otimes_R K_n, \mathcal{F}))$$

is an isomorphism for all $n \in \mathbb{Z}$. This shows that the canonical morphism

$$H_n(\underline{\mathrm{Hom}}(K_2 \otimes_R \ldots \otimes_R K_n, \mathcal{F})) \to H_n(\underline{\mathrm{Hom}}(K_2 \otimes_R \ldots \otimes_R K_n, \mathcal{F})) \otimes_R R_1$$

is an isomorphism and, since R_1 is a flat R-module, we get that the canonical morphism

$$\underline{\mathrm{Hom}}(K_2 \otimes_R \ldots \otimes_R K_n, \mathcal{F}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(K_2 \otimes_R \ldots \otimes_R K_n, \mathcal{F}) \otimes_R R_1 \qquad (1)$$

is a quasi-isomorphism. By assumption, we have that $j^*(\mathcal{F}) \simeq 0$, which in particular implies that $\mathcal{F} \otimes_R R_1 \simeq 0$. Now, since $K_2 \otimes_R \ldots \otimes_R K_n$ is a bounded complex of projective modules, we get an isomorphism

$$\underline{\mathrm{Hom}}(K_2 \otimes_R \ldots \otimes_R K_n, \mathcal{F}) \otimes_R R_1 \cong \underline{\mathrm{Hom}}(K_2 \otimes_R \ldots \otimes_R K_n, \mathcal{F} \otimes_R R_1) \simeq 0. \qquad (2)$$

Finally, combining (1) and (2), we get

$$\underline{\mathrm{Hom}}(K_2 \otimes_R \ldots \otimes_R K_n, \mathcal{F}) \simeq 0.$$

After applying $\mathrm{map}_{\mathbf{D}(R)}(R, -)$, it follows that

$$\mathrm{map}_{\mathbf{D}(X)}(K_2, \underline{\mathrm{Hom}}(K_3 \otimes_R \ldots \otimes_R K_n, \mathcal{F})) \simeq 0.$$

We can repeat the above argument until we get that $\mathrm{map}_{\mathbf{D}(X)}(K_n, \mathcal{F}) \simeq 0$, which then implies that

$$\mathcal{F} \simeq \mathcal{F} \otimes_R R_n \simeq 0,$$

as desired. □

We will now generalize this to non-affine schemes:

Theorem 4.45. *Let X be a quasi-compact and quasi-separated scheme and let $j \colon U \to X$ be a quasi-compact open immersion. Then the sequence*

$$\mathbf{D}(X)_U \xrightarrow{i} \mathbf{D}(X) \xrightarrow{j^*} \mathbf{D}(U)$$

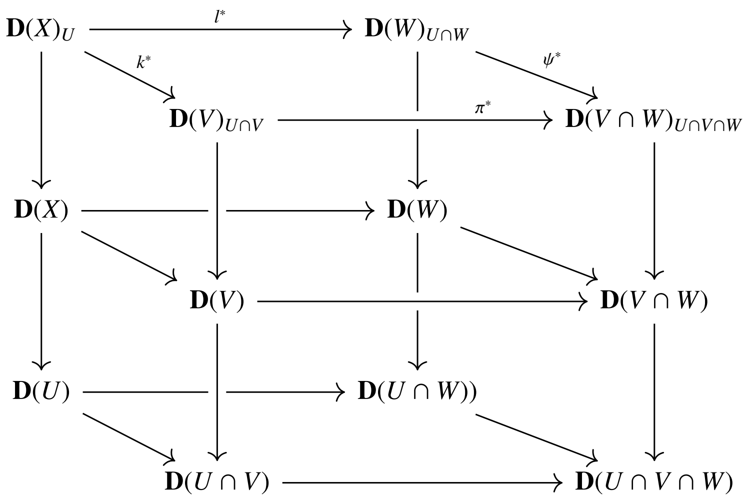
is exact, all three categories are compactly generated and both functors preserve compact objects.

Proof: Let us start with the affine case $X = \mathrm{Spec}(A)$. It is clear that the above sequence is exact. By Proposition 4.44, we know that $\mathbf{D}(X)$ and $\mathbf{D}(X)_U$ are compactly generated. Since j^* has a fully faithful right adjoint, it follows that $\mathbf{D}(U)$ is compactly generated as well. Furthermore, the morphism j^* preserves compact objects by Proposition 4.32. Now we observe that the right adjoint of the inclusion i is given by

$$\begin{aligned} G\colon \mathbf{D}(X) &\to \mathbf{D}(X)_U \\ \mathcal{F} &\mapsto \mathrm{fib}(\eta\colon \mathcal{F} \to j_* j^* \mathcal{F}) \end{aligned}$$

and, since both j_* and j^* preserve colimits, it follows that G preserves colimits as well. Thus i preserves compact objects, which completes the proof in the affine case.

We now turn towards the general case. Using the same arguments as above, the only thing left to show is that $\mathbf{D}(X)_U$ is compactly generated (note that we may pick $U = \emptyset$). We will inductively show that $\mathbf{D}(X)_U$ is generated by a single object. By the usual induction argument, we may assume that we can write $X = V \cup W$ where $V = \mathrm{Spec}(R)$ is affine and the statement of the theorem holds for W. The situation may be depicted in the following diagram:



where all horizontal squares are pullback squares. Let us by $\ker \pi^*$ denote the full subcategory of $\mathbf{D}(V)_{U\cap V}$ spanned by all objects that become 0 after applying π^*. We observe that, by Zariski descent, we have $\ker \pi^* = \mathbf{D}(V)_{(U\cap V)\cup(V\cap W)}$ and thus $\ker \pi^*$ is compactly generated by a single element. Now let us consider the sequence

$$\ker \pi^* \hookrightarrow \mathbf{D}(V)_{U\cap V} \xrightarrow{\pi^*} \mathbf{D}(V \cap W)_{U\cap VW}. \tag{1}$$

It is easy to see that this is in fact an exact sequence of presentable stable ∞-categories. By Theorem 4.30, it follows that the right adjoint of

$$\mathbf{D}(V) \to \mathbf{D}(V \cap W)$$

restricts to a right adjoint of π^*. In particular, the right adjoint of π^* preserves all colimits and thus π^* preserves compact objects. Like in the affine case, it follows that the inclusion $\ker \pi^* \hookrightarrow \mathbf{D}(V)_{U\cap V}$ preserves compact objects as well. So we have shown that we may apply Corollary 3.26 to the sequence (1).

We will now construct a compact generator of $\mathbf{D}(X)_U$: Recall that the square

$$\begin{array}{ccc} \mathbf{D}(X)_U & \xrightarrow{l^*} & \mathbf{D}(W)_{U\cap W} \\ \downarrow{\scriptstyle k^*} & & \downarrow{\scriptstyle \psi^*} \\ \mathbf{D}(V)_{U\cap V} & \xrightarrow{\pi^*} & \mathbf{D}(V\cap W)_{U\cap V\cap W} \end{array}$$

is a pullback square. Now let γ be a compact generator of $\ker \pi^*$. Then the tuple

$$(\gamma, 0, \pi^*\gamma \xrightarrow{\simeq} 0)$$

gives rise to an element in $\mathbf{D}(X)_U$, which we will denote by Q_1. Let $K_W \in \mathbf{D}(W)_{U\cap W}$ be a compact generator and observe that $\psi^*(K_W) \in \mathbf{D}(V\cap W)_{U\cap V\cap W}$ is again compact. But we have seen above that we may apply Corollary 3.26 to the sequence (1) in order to conclude that there is some $\alpha_V \in \mathbf{D}(V)_{U\cap V}$, as well as an equivalence

$$\beta\colon \pi^*(\alpha_V) \xrightarrow{\simeq} \psi^*(K_W) \oplus \Sigma\psi^*(K_W).$$

Thus the tuple

$$(\alpha_V, K_W \oplus \Sigma K_W, \beta)$$

gives rise to an element Q_2 in $\mathbf{D}(X)_u$. We now define $Q = Q_1 \oplus Q_2$ and claim that this is a compact generator of $\mathbf{D}(X)_U$. It is clear that Q is compact, so let us pick $\mathcal{F} \in \mathbf{D}(X)_U$ such that

$$\mathrm{map}_{\mathbf{D}(X)_U}(\Sigma^n Q, \mathcal{F}) \simeq 0$$

for all $n \in \mathbb{Z}$. In particular, it follows that

$$\mathrm{map}_{\mathbf{D}(V)_{U\cap V}}(\Sigma^n \gamma, \mathcal{F}|_V) \simeq 0 \tag{2}$$

for all $n \in \mathbb{Z}$. Consider the unit map $\eta\colon \mathcal{F}|_V \to \pi_*\pi^* \mathcal{F}|_V$ and let M denote the cofiber. Note that $M \in \ker \pi^*$. Furthermore, the equivalence (2) implies that

$$\mathrm{map}_{\mathbf{D}(V)_{U\cap V}}(\Sigma^n \gamma, M) \simeq 0,$$

thus $M \simeq 0$ and therefore $\mathcal{F}|_V \simeq \pi_*\pi^* \mathcal{F}|_V \simeq \pi_*\psi^* \mathcal{F}|_W$. So it suffices to see that $\mathcal{F}|_W \simeq 0$. But, by assumption,

$$0 \simeq \mathrm{map}_{\mathbf{D}(W)_{W\cap V}}(\Sigma^n K_W \oplus \Sigma^{n+1} K_W, \mathcal{F}|_W),$$

which implies that $\mathcal{F}|_W \simeq 0$, as K_W is a compact generator. This completes the proof. □

Corollary 4.46. *The induced exact sequence*

$$\mathrm{Perf}(X)_U \to \mathrm{Perf}(X) \to \mathrm{Perf}(U)$$

is an exact sequence of small idempotent complete stable ∞-categories. Thus the induced sequence

$$\mathbb{K}(X)_U \to \mathbb{K}(X) \to \mathbb{K}(U)$$

is a cofiber sequence.

Proof: By Theorem 4.45, we may apply Proposition 3.23. The second part of the claim is then clear since non-connective K-theory is localizing. □

4.3. Nisnevich Descent for Algebraic K-Theory

In order to finally conclude that K-theory is a Nisnevich sheaf, we need one more input:

Theorem 4.47. *Let S be a quasi-compact and quasi-separated scheme and let*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

be an elementary distinguished Nisnevich square in $\mathrm{Sm}_{/S}$. Then the restriction of p^* induces an equivalence of ∞-categories

$$\mathbf{D}(X)_U \xrightarrow{\simeq} \mathbf{D}(V)_{U\times_X V}$$

and thus, in particular, an equivalence $\mathrm{Perf}(X)_U \xrightarrow{\simeq} \mathrm{Perf}(V)_{U\times_X V}$.

Proof: Observe that, by flat base change, the right adjoint $p_*\colon \mathbf{D}(V) \to \mathbf{D}(X)$ restricts to a functor

$$\mathbf{D}(V)_{U\times_X V} \to \mathbf{D}(X)_U.$$

Therefore it suffices to see that the unit and counit map

$$\eta\colon \mathcal{F} \to p_*p^*\mathcal{F} \qquad \text{and} \qquad \varepsilon\colon p^*p_*\mathcal{G} \to \mathcal{G}$$

are invertible, for $\mathcal{F} \in \mathbf{D}(X)_U$ and $\mathcal{G} \in \mathbf{D}(V)_{U\times_X V}$. Let us start by showing that the unit map is invertible:

For this, we observe that, by Zariski descent and flat base change, we may assume that $X = \mathrm{Spec}\,(A)$ is affine. Since $\mathbf{D}(X)_U$ is compactly generated by $\mathrm{Perf}(X)_U$ and since all objects in $\mathrm{Perf}(X)_U$ are bounded, we can write $\mathcal{F}$ as a colimit of bounded complexes. As the functors p_* and p^* both preserve colimits, we may thus assume that $\mathcal{F}$ is bounded. Since p_* and p^*

commute with shifts, we may assume that there is an $n \in \mathbb{N}$ such that $H_i(\mathcal{F}) = 0$ for all $i < 0$ and $i > n$. Now, for all $k \leq n$, there is a canonical fiber sequence

$$H_k(\mathcal{F})[k] \to \tau_{\leq k}\mathcal{F} \to \tau_{\leq k-1}\mathcal{F}.$$

Thus we may use induction to reduce to the case where $\mathcal{F} = M[0]$ for some A-module M. Furthermore, we can write M as the union of its finitely generated submodules to assume that it is finitely generated. Let us now pick $f_1, \ldots, f_n \in A$ such that

$$U = \mathrm{Spec}\,(A_1) \cup \ldots \cup \mathrm{Spec}\,(A_n),$$

where $A_i = A[f_i^{-1}]$. Let us write $I = (f_1, \ldots, f_n)$, $Z = \mathrm{Spec}\,(R/I)$ and

$$i\colon\ \mathrm{Spec}\,(R/I) \to \mathrm{Spec}\,(R)$$

for the corresponding closed immersion. The assumption that $M|_U = 0$ now precisely says that $M \otimes_A A[f_i^{-1}] = 0$ for all i. Since M is finitely generated, we may thus find an $N \in \mathbb{N}$ such that $I^N M = 0$. Now, for all $1 \leq k < N$, there is an exact sequence

$$0 \to I^k M \to I^{k-1} M \to I^{k-1} M / I^k M \to 0$$

and we may use induction to the reduce to the case where $IM = 0$. In other words, we may assume that there is some R/I-module M' such that $M = i_* M'$. For the rest of the proof, let us use notation as in the following diagram:

$$\begin{array}{ccccc} U \times_X V & \longrightarrow & V & \xleftarrow[i']{} & V \times_X Z \\ \downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ U & \xrightarrow{j} & X & \xleftarrow[i]{} & Z \end{array}$$

Here, the left square is a Nisnevich square by assumption, and therefore p' is an isomorphism. Now, using flat base change, we compute:

$$M \simeq i_* M' \simeq i_* p'_* p'^* M' \simeq p_* i'_* p'^* M' \simeq p_* p^* i_* M' \simeq p_* p^* M.$$

It is easy to see that this equivalence agrees with the unit map.

It remains to show that the counit map is an equivalence. For this, we observe that we can use exactly the same methods as above to show that, for an affine open $j\colon\ \mathrm{Spec}\,(A) \to V$ and any $\mathcal{F} \in \mathbf{D}(A)$ with $\mathcal{F}|_{(U \times_X V) \cap \mathrm{Spec}(A)} \simeq 0$, the counit map

$$j^* p^* p_* j_* \mathcal{F} \to \mathcal{F}$$

is an equivalence. Since we can cover X by finitely many affine opens, we can then proceed by induction over the size of the cover and Zariski descent, as usual. This completes the proof. □

Theorem 4.48. *Let S be a quasi-compact and quasi-separated scheme. Then the presheaf*

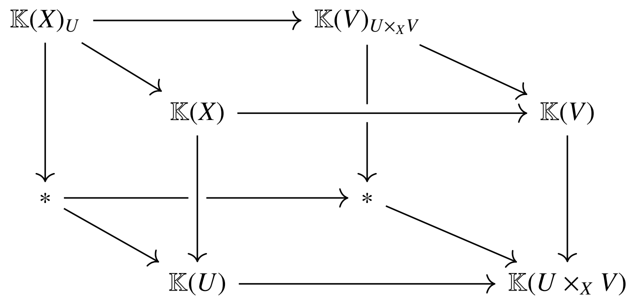
$$K(-)\colon\ \mathrm{Sm}_{/S} \to \mathcal{S}$$

is a Nisnevich sheaf.

Proof: By Theorem 1.15, it suffices to see that, for any elementary distinguished Nisnevich square

$$\begin{CD} U \times_X V @>>> V \\ @VVV @VV{p}V \\ U @>{j}>> X \end{CD}$$

the induced square of K-theory spaces is a pullback square. Let us consider the induced cube



By Corollary 4.46, the left and right side are pushout squares. The map $\mathbb{K}(X)_U \to \mathbb{K}(V)_{U \times_X V}$ in the back is induced by the equivalence $\mathrm{Perf}(X)_U \to \mathrm{Perf}(V)_{U \times_X V}$ from Theorem 4.47 and is thus an equivalence. It follows that the back square is a pushout square as well. Therefore the front square is a pushout square, too and thus also a pullback square, as Sp is stable. Since $\Omega^\infty\colon \mathrm{Sp} \to \mathcal{S}$ preserves finite limits, the claim follows. □

5. $\mathbb{A}^1$-Invariance of Algebraic K-Theory

In order to conclude that algebraic K-theory is represented by a motivic space, we would like to see that the functor $K\colon \mathrm{Sm}_{/S} \to \mathcal{S}$ is $\mathbb{A}^1$-invariant. However, in general this is not true (see Example 5.18). In this section we will introduce the G-Theory of a noetherian scheme X and see that it agrees with K-theory if X is regular (Corollary 5.14). Then Quillen's classical result (Theorem 5.15) about the $\mathbb{A}^1$-invariance of G-theory implies that algebraic K-theory is $\mathbb{A}^1$-invariant for regular and noetherian schemes.

Definition 5.1. Let R be a noetherian ring. We call a complex $\mathcal{F} \in \mathbf{D}(R)$ *coherent* if it is homologically bounded and if $H_i(\mathcal{F})$ is a finitely generated R-module for all i. We will write $\mathbf{D}_{\mathrm{coh}}(R)$ for the full subcategory of $\mathbf{D}(R)$ spanned by the coherent objects.

Remark 5.2. Note that the inclusion $\mathbf{D}_{\mathrm{coh}}(R) \hookrightarrow \mathbf{D}(R)$ preserves cofibers and shifts and thus $\mathbf{D}_{\mathrm{coh}}(R)$ is stable. Also note that $\mathbf{D}_{\mathrm{coh}}(R)$ is essentially small. Furthermore, retracts of finitely generated modules are still finitely generated and thus $\mathbf{D}_{\mathrm{coh}}(R)$ is idempotent complete as well.

We will now globalize this definition to noetherian schemes:

Definition 5.3. Let X be a noetherian scheme. We will call an object $\mathcal{F} \in \mathbf{D}(X)$ *coherent* if, for any affine open $j\colon \mathrm{Spec}(A) \hookrightarrow X$, the restriction $j^*\mathcal{F}$ is coherent in $\mathbf{D}(A)$. We will write $\mathbf{D}_{\mathrm{coh}}(X)$ for the full subcategory spanned by the coherent objects.

Remark 5.4. Note that, for an open immersion of affine schemes $f\colon \operatorname{Spec}(A) \hookrightarrow \operatorname{Spec}(R)$, the functor $f^*\colon \mathbf{D}(R) \to \mathbf{D}(A)$ preserves coherent objects. Thus, for an affine scheme $X = \operatorname{Spec}(R)$, the two definitions of $\mathbf{D}_{\mathrm{coh}}(X)$ above agree.

Remark 5.5. Note that, if $j\colon U \hookrightarrow X$ is an open immersion, then the functor $j^*\colon \mathbf{D}(X) \to \mathbf{D}(U)$ restricts to a functor $\mathbf{D}_{\mathrm{coh}}(X) \to \mathbf{D}_{\mathrm{coh}}(U)$ by definition.

5.6. Recall that the *small Zariski site* X_{Zar} is given by restricting the big Zariski site $\mathrm{Sch}_{/X}$ to the full subcategory spanned by the open subschemes of X.

Proposition 5.7. *The functor* $\mathbf{D}_{\mathrm{coh}}\colon X_{\mathrm{Zar}}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ *is a sheaf with respect to the Zariski topology.*

Proof: Since the subcategory of affine open subschemes defines a basis for the topology on X_{Zar}, it suffices to see that, for any affine open $\operatorname{Spec}(A)$ and for any two open immersions $j_1\colon \operatorname{Spec}(B_1) \to \operatorname{Spec}(A)$ and $j_2\colon \operatorname{Spec}(B_2) \to \operatorname{Spec}(A)$ with $\operatorname{Spec}(A) = \operatorname{Spec}(B_1) \cup \operatorname{Spec}(B_2)$, the induced square

$$\begin{array}{ccc} \mathbf{D}_{\mathrm{coh}}(A) & \xrightarrow{j_2^*} & \mathbf{D}_{\mathrm{coh}}(B_2) \\ \downarrow{\scriptstyle j_1^*} & & \downarrow \\ \mathbf{D}_{\mathrm{coh}}(B_1) & \longrightarrow & \mathbf{D}_{\mathrm{coh}}(B_1 \otimes_A B_2) \end{array}$$

is a pullback square. In light of Proposition 4.13, this boils down to showing that an object $\mathcal{F} \in \mathbf{D}(A)$ is coherent if and only if $j_1^*\mathcal{F}$ and $j_2^*\mathcal{F}$ are coherent. But this follows since being zero and being finitely generated are both local properties (see [Sta20, Tag 00EO]). □

Remark 5.8. It follows that the functor $\mathbf{D}_{\mathrm{coh}}\colon X_{\mathrm{Zar}}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is the right Kan-extension of its restriction to the full subcategory spanned by all affine open subschemes $X_{\mathrm{Zar}}^{\mathrm{aff\ op}}$. Since the inclusion

$$\mathrm{Cat}_\infty^{\mathrm{Perf}} \to \mathrm{Cat}_\infty$$

preserves limits by Remark 3.13 and [Lur12, Theorem 1.1.4.4.], it follows from Remark 5.2 that $\mathbf{D}_{\mathrm{coh}}(X)$ is a small idempotent complete stable ∞-category.

Definition 5.9. We define the *G-theory space* $G(X)$ of X to be the K-theory space associated to the stable ∞-category $\mathbf{D}_{\mathrm{coh}}(X)$.

5.10. Note that, for a flat morphism $f\colon X \to Y$, the induced functor

$$f^*\colon \mathbf{D}(Y) \to \mathbf{D}(X)$$

restricts to a functor

$$\mathbf{D}_{\mathrm{coh}}(f)\colon \mathbf{D}_{\mathrm{coh}}(Y) \to \mathbf{D}_{\mathrm{coh}}(X)$$

and thus we get an induced morphism

$$G(f)\colon G(Y) \to G(X).$$

5.11. By definition there is a fully faithful natural inclusion functor

$$\mathrm{Perf}(X) \hookrightarrow \mathbf{D}_{\mathrm{coh}}(X). \tag{1}$$

inducing a natural map

$$K(X) \to G(X).$$

However (1), is in general not an equivalence: Consider for example $X = \mathrm{Spec}\,(\mathbb{Z}/4\mathbb{Z})$ and the coherent object $\mathbb{Z}/2\mathbb{Z}[0]$ in $\mathbf{D}_{\mathrm{coh}}(\mathbb{Z}/4\mathbb{Z})$ given by the module $\mathbb{Z}/2\mathbb{Z}$ concentrated in degree zero. Since the projective dimension of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}/4\mathbb{Z}$ is infinite, the module $\mathbb{Z}/2\mathbb{Z}$ does not admit a finite projective resolution. In other words, the object $\mathbb{Z}/2\mathbb{Z}[0]$ is not equivalent to an object in $\mathrm{Perf}(X)$.

The problem is precisely that $X = \mathrm{Spec}\,(\mathbb{Z}/4\mathbb{Z})$ is *not regular*, which (in the noetherian case) implies the existence of finitely generated modules that do not admit a finite projective resolution. We will now show that this is the only issue that can appear, i.e. that the above functor is an equivalence if X is regular.

5.12. Recall that a noetherian local ring $(R, \mathfrak{m}, \kappa)$ of finite Krull dimension is called regular if the Krull dimension of R agrees with $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$. By a famous theorem of Serre, a noetherian local ring R is regular if and only if it has finite global dimension, i.e. there is a natural number n such that every R-module M has a projective resolution of length n.

A general noetherian ring is called regular if, for every prime ideal $\mathfrak{p} \subseteq A$, the localization $A_\mathfrak{p}$ is a regular local ring. Similarly, a noetherian scheme X is called regular if $\mathcal{O}_{X,x}$ is regular for all $x \in X$.

Proposition 5.13. *Let X be a regular noetherian scheme. Then the inclusion*

$$\mathrm{Perf}(X) \to \mathbf{D}_{\mathrm{coh}}(X)$$

is an equivalence

Proof: Since both $\mathrm{Perf}(-)$ and $\mathbf{D}_{\mathrm{coh}}(-)$ are Zariski sheaves, we may immediately reduce to $X = \mathrm{Spec}\,(R)$ for some regular ring R. We show that any finitely generated R-module has a finite resolution by finitely generated projective R-modules and after that, the proposition follows from an easy homological algebra argument. So let M be a finitely generated projective R-module. Since R is noetherian, we may find a projective resolution

$$\ldots \to P_n \to \ldots \to P_1 \to P_0 \twoheadrightarrow M,$$

where all P_i are finitely generated. Let $x \in \mathrm{Spec}\,(R)$ be a point and let $i = \dim(\mathcal{O}_{X,x})$. Then, by Serre's theorem, the global dimension of $\mathcal{O}_{X,x}$ is i and thus $\ker(P_j \to P_{j-1}) \otimes_R \mathcal{O}_{X,x}$ is a finitely generated projective and hence finite free $\mathcal{O}_{X,x}$-module for all $j \geq i$. Therefore there is an affine open neighbourhood $U \subseteq X$ of x such that $\ker(P_j \to P_{j-1})\big|_U$ is finite free for all $j \geq i$. As R is noetherian and thus in particular quasi-compact, it follows that there is some $N \gg 0$ such that $\ker(P_N \to P_{N-1})$ is locally free and thus

$$\ldots 0 \to \ker(P_N \to P_{N-1}) \hookrightarrow P_N \to \ldots \to P_0 \twoheadrightarrow M$$

is the desired finite projective resolution. □

As a consequence we obtain:

Corollary 5.14. *Let X be a regular noetherian scheme. Then the canonical map*

$$K(X) \to G(X)$$

is an equivalence.

A fundamental theorem of Quillen in [Qui73, §7 Proposition 4.1] says that G-theory is $\mathbb{A}^1$-invariant:

Theorem 5.15. *Let X be a noetherian scheme. Then the canonical map $X \times \mathbb{A}^1 \to X$ induces an equivalence*

$$G(X) \to G(X \times \mathbb{A}^1).$$

If X is regular, it follows that the induced map

$$K(X) \to K(X \times \mathbb{A}^1)$$

is an equivalence.

Combining this with Theorem 4.48, we finally reach the following result:

Theorem 5.16. *Let X be a regular noetherian scheme. Then the algebraic K-theory functor*

$$K\colon \mathrm{Sm}_{/X} \to \mathcal{S}$$

is a motivic space.

5.17. Of course, the proof of Theorem 5.15 in [Qui73] uses a construction of the algebraic K-theory of a scheme which is different from ours. In order to justify our use of this theorem, we will now quickly discuss why our definition of algebraic K-theory agrees with the definition used in [Qui73].

First of all, it is already noted in [TT07, Proposition 3.10] that Quillen's definition of the algebraic K-theory of a noetherian scheme agrees with the definition given by Thomason and Trobaugh. Thus it suffices to compare our definition of algebraic K-theory with the definition given in [TT07]. But under the light of Remark 4.25, this follows from [BGT13, Theorem 7.8].

Example 5.18. We conclude this chapter by giving an example that shows that Theorem 5.16 fails if X is not regular. For this, let k be a field and let us consider the affine scheme $X = \mathrm{Spec}\left(k[X]/(X^2)\right) = \mathrm{Spec}\,(k[\varepsilon])$. We will show that the canonical morphism

$$K_1(k[\varepsilon][T]) \to K_1(k[\varepsilon])$$

induced by the ring homomorphism $k[\varepsilon][T] \to k[\varepsilon]$ sending T to 0 is *not* an isomorphism. For this, we recall that, for a ring R, the first K-group $K_1(R)$ can be computed as the abelianization of the general linear group GL(R). Since $k[\varepsilon]$ is a local ring, it follows that the canonical morphism

$$K_1(k[\varepsilon]) \to k[\varepsilon]^\times$$

given by the determinant is an isomorphism (see [Wei13, Lemma III.1.4]). Furthermore, we have a commutative diagram

$$\begin{array}{ccc} k[\varepsilon][T]^\times = GL_1(k[\varepsilon][T]) & & \\ \big\downarrow & \searrow & \\ K_1(K[\varepsilon][T]) & \longrightarrow & K_1(k[\varepsilon]) = k[\varepsilon]^\times \end{array}$$

where the left vertical arrow is injective (it has a left inverse given by the determinant). But the horizontal arrow cannot be injective, as the map

$$k[\varepsilon][T]^\times \to k[\varepsilon]^\times$$

is not injective. For example, the unit $1 - \varepsilon T$ is sent to 1.

Appendix

A. Sheaf Theory

In this section we will recall basic notions of sheaf theory that we are using throughout this thesis.

Definition A.1.

i) Let C be a 1-category. For $c \in C$, a *sieve* on c is given by a subobject of the presheaf $h(c)$.

ii) Let $c \in C$ and let R be a sieve on c. Let $f: d \to c$ be a morphism. Then the *pullback sieve* f^*R is the subobject of $h(d)$ given by the assignment:

$$k \mapsto \{\alpha: k \to d \text{ such that } f \circ \alpha \text{ is in R}\}.$$

Definition A.2. Let C be a 1-category. A *Grothendieck-topology* τ on C is given by a collection of sieves τ_c on every $c \in C$, so called *covering sieves*, that satisfy the following:

– For every $c \in C$, the maximal sieve $\mathrm{id}_c: h(c) \to h(c)$ is a covering sieve.

– For every $c \in C$, every covering sieve R on c and every morphism $f: d \to c$, the pullback sieve f^*R is a covering sieve on d.

– Let R be a sieve on $c \in C$ and let S be a covering sieve on c. If, for all $f: d \to c$ in S, the sieve f^*R is covering d, then R is itself covering.

A pair (C, τ) consisting of a 1-category C and a topology τ on C, will be called a *site*.

Usually, Grothendieck-topologies arise via so called Grothendieck-pretopologies:

Definition A.3. Let C be a 1-category with pullbacks. Then a *pretopology* τ on C assigns to each object $c \in C$ a collection τ_c of families of morphisms $\{x_i \to c\}$, called *covering families* that satisfy the following conditions:

– Every familiy consisting of a single isomorphism $\alpha: x \xrightarrow{\cong} c$ is covering c.

– If $S = \{x_i \to c\}$ is covering c and if $g: d \to c$ is any morphism, then the family $g^*S = \{x_i \times_c d \to d\}$ of the pulled back morphisms is covering d.

– If $S = \{x_i \to c\}$ is any family and $\{y_j \to c\}$ is a covering familiy such that for all j the family $\{x_i \times_c y_j \to y_j\}$ is covering y_j, then S is itself a covering family.

The following is an easy excercise:

Construction A.4. Let C be a 1-category equipped with a pretopology τ'. Then we define a topology τ on C by declaring that a sieve R on c is covering if and only if it contains a family of morphisms that is an element of τ'_c. It is easy to check that this indeed defines a topology on C.

Construction A.5. Let (C, τ) be a site. Consider the collection W consisting of all morphisms of the form $R \hookrightarrow h(c)$ with $R \in \tau_c$. We define the category of *τ-sheaves of spaces on C*, denoted by $\mathrm{Sh}_\tau(C)$, to be the full subcategory spanned by the W-local objects of $\mathrm{Psh}(C)$ (see [Lur09, §5.5.4]). In particular, we have a localization functor $L_{(C,\tau)}\colon \mathrm{Psh}(C) \to \mathrm{Sh}_\tau(C)$ which is left adjoint to the inclusion $i\colon \mathrm{Sh}_\tau(C) \to \mathrm{Psh}(C)$. We will call a morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ a *τ-local equivalence*, if $L_{(C,\tau)}(\alpha)$ is an equivalence.

The main result is the following ([Lur09, Proposition 6.2.2.7]:

Theorem A.6. *In the above situation, the functor $L_{(C,\tau)}\colon \mathrm{Psh}(C) \to \mathrm{Sh}_\tau(C)$ preserves finite limits. In particular, the category $\mathrm{Sh}_\tau(C)$ is an ∞-topos.*

There is also a notion weaker than the one of a pretopology, which can be used to construct categories of sheaves:

Definition A.7. Let C be a 1-category with pullbacks. Then a *coverage* σ on C assigns to each object $c \in C$ a collection σ_c of families of morphisms $\{x_i \to c\}$, called *covering families*, that satisfy the following condition:
If $\{x_i \to c\}$ is covering c and $g\colon d \to c$ is any morphism, then the family $\{x_i \times_c d \to d\}$ of the pulled back morphisms is covering d.

Definition A.8. Let C be a 1-category with pullbacks.

- Let $S = \{f_i\colon x_i \to c\}$ be a family of morphisms. The *sieve generated by S* is the sieve given by the assignment:

$$x \mapsto \{f\colon x \to c \text{ such that } f \text{ factors through one of the } f_i\}\,.$$

- Let σ be a coverage on C. We say that a presheaf $\mathcal{F} \in \mathrm{Psh}(C)$ is a *Čech-sheaf for σ* if, for all $c \in C$ and $S = \{x_i \to c\}$ in σ_c, the canonical morphism

$$\mathcal{F}(c) \to \mathrm{map}_{\mathrm{Psh}(C)}(R, \mathcal{F})$$

is an equivalence, where R denotes the sieve generated by the family S.

Construction A.9. Let C be a 1-category with finite limits and let $f\colon c \to d$ be a morphism in C. The *Čech-nerve of f* is the simplicial object $\check{C}_\bullet(f)$ given by the $(k+1)$-fold fiber product of c over d with itself

$$\check{C}_\bullet(f) = \Big(\ldots c \times_d c \times_d c \mathrel{\substack{\rightarrow\\ \rightarrow\\ \rightarrow}} c \times_d c \rightrightarrows c\Big),$$

where the face maps are the projections and the degeneracies are the canonical maps induced by the universal property of the fiber product. Furthermore, this simplicial object comes with a canonical augmentation $\check{C}_\bullet(f) \to d$.

Now, let $\{f_i\colon x_i \to c\}$ be a family of morphisms in C and let us by R denote the sieve generated by this family. We get an induced map

$$f\colon \coprod_i h(x_i) \to h(c)$$

in $\mathrm{Fun}(C, \mathrm{Set}) \subseteq \mathrm{Psh}(C)$ and thus a simplicial object $\check{C}_\bullet(f)$ in $\mathrm{Fun}(C, \mathrm{Set})$, which we may also consider as a simplicial object in $\mathrm{Psh}(C)$. Furthermore, the augmentation factors through the subobject R and we get an induced morphism

$$\phi_f \colon |\check{C}_\bullet(f)| \to R$$

in $\mathrm{Psh}(C)$.

Lemma A.10. *In the situation of Construction A.9, the morphism ϕ_f is an equivalence.*

Proof: By [Lur09, Proposition 6.2.3.4], both $|\check{C}_\bullet(f)|$ and R are subobjects of the 0-truncated object $h(d)$ and therefore 0-truncated themselves. So it suffices to show that the 0-truncation $\tau_{\leq 0}(\phi_f)$ is an equivalence. Since $\tau_{\leq 0}$ commutes with colimits, it suffices to show that the canonical morphism

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}} \check{C}_\bullet(f) \to R$$

is an isomorphism, where the colimit is the 1-colimit taken in the 1-category $\mathrm{Fun}(C, \mathrm{Set})$. Let J be the "reflexive coequalizer-category", i.e. the full subcategory of Δ^{op} spanned by the two objects $[0]$ and $[1]$. Then the canonical inclusion $J \to \Delta^{\mathrm{op}}$ is a colimit-cofinal 1-functor and it follows that $\mathrm{colim}_{\Delta^{\mathrm{op}}} \check{C}_\bullet(f)$ is given by the reflexive coequalizer of the diagram

$$\Big(\coprod_i h(x_i)\Big) \times_{h(c)} \Big(\coprod_i h(x_i)\Big) \rightrightarrows \coprod_i h(x_i).$$

Now, one can easily and explicitly check that the canonical map from this coequalizer to R is an isomorphism. □

Remark A.11. For two Grothendieck-topologies τ and τ' on a 1-category C, we define the intersection $\tau \cap \tau'$ by forming the intersection $\tau_c \cap \tau'_c$ for every $c \in C$. This clearly is a Grothendieck-topology again. So we may speak of the smallest Grothendieck-topology containing a certain collection of sieves.

Definition A.12. Let C be a 1-category with pullbacks and σ a coverage on C. Then we define the *Grothendieck-topology generated by σ* to be the smallest Grothendieck-topology containing, for all $c \in C$, all sieves generated by families $\{x_i \to c\}$ that lie in σ_c. We denote this topology by $\overline{\sigma}$.

We will now relate the two notions of sheaves introduced above:

Proposition A.13. *Let C be a 1-category with finite limits and let σ be a coverage on C. Then a presheaf $\mathcal{F} \in \mathrm{Psh}(C)$ is a Čech-sheaf with respect to σ if and only if it is a sheaf with respect to $\overline{\sigma}$.*

Proof: It is clear that every $\overline{\sigma}$-sheaf is a σ-Čech-sheaf. For the converse, let us consider the assignment Ψ that assigns to every $c \in C$ the collection Ψ_c of all sieves $R \hookrightarrow h(c)$ such that the following holds: For every $f\colon x \to c$ and every σ-Čech-sheaf E, the canonical morphism

$$E(x) \to \mathrm{map}_{\mathrm{Psh}(C)}\left(R \times_{h(c)} h(x), E\right)$$

is an equivalence. If we can show that Ψ is a topology on C, we are done: In this case, the topology Ψ clearly contains the coverage σ and therefore also $\overline{\sigma}$. So, if $\mathcal{F}$ is Čech-sheaf, it is a Ψ-sheaf by construction and thus a $\overline{\sigma}$-sheaf and the Proposition follows.

Therefore, we only have to check that Ψ satisfies the three axioms of Definition A.2. The first two are obviously satisfied. For the third, let $c \in C$ and let $R \hookrightarrow h(c)$ be any sieve. Let $S \hookrightarrow h(c)$ be a sieve in Ψ_c such that, for every $x \to c$ in S, the pulled back sieve

$$R \times_{h(c)} h(x) \to h(x)$$

is in Ψ_x. Let E be any σ-Čech-sheaf. We have to show that, for any $f\colon y \to c$, the canonical map

$$E(y) \to \mathrm{map}_{\mathrm{Psh}(C)}(R \times_{h(c)} h(y), E)$$

is an equivalence. Consider the commutative diagram

$$\begin{array}{ccc} E(y) & \longrightarrow & \mathrm{map}_{\mathrm{Psh}(C)}(R \times_{h(c)} h(y), E) \\ \downarrow & & \downarrow \\ \mathrm{map}_{\mathrm{Psh}(C)}(S \times_{h(c)} h(y), E) & \longrightarrow & \mathrm{map}_{\mathrm{Psh}(C)}(R \times_{h(c)} S \times_{h(c)} h(y), E) \end{array}$$

We know that the left vertical map is an equivalence, as S is in Ψ_c. Writing R as a colimit of representables, we get that the right vertical map is an equivalence as well. So it suffices to see that the the lower horizontal morphism is an equivalence. Now, we use that

$$S \simeq \operatorname*{colim}_{h(k)\to S} h(k) \simeq \operatorname*{colim}_{k\to c \text{ in } S} h(k).$$

So we get a commutative diagram

$$\begin{array}{ccc} \mathrm{map}_{\mathrm{Psh}(C)}(S \times_{h(c)} h(y), E) & \longrightarrow & \mathrm{map}_{\mathrm{Psh}(C)}(R \times_{h(c)} S \times_{h(c)} h(y), E) \\ \downarrow & & \downarrow \\ \lim\limits_{k\to c \text{ in } S} \mathrm{map}_{\mathrm{Psh}(C)}(h(k) \times_{h(c)} h(y), E) & \longrightarrow & \lim\limits_{k\to c \text{ in } S} \mathrm{map}_{\mathrm{Psh}(C)}(R \times_{h(c)} h(k) \times_{h(c)} h(y), E) \end{array}$$

where the vertical arrows are equivalences, by the universality of colimits in $\mathrm{Psh}(C)$. So it suffices to see that the bottom horizontal arrow is an equivalence. But, for every $k \to x$ in S, the sieve

$$R \times_{h(c)} h(k) \to h(k)$$

is in Ψ_k and thus it follows that the morphism

$$E(k \times_c y) \simeq \mathrm{map}_{\mathrm{Psh}(C)}(h(k) \times_{h(c)} h(y), E) \to \mathrm{map}_{\mathrm{Psh}(C)}(R \times_{h(c)} h(k) \times_{h(c)} h(y), E)$$

is an equivalence. This completes the proof. □

Remark A.14. Let C be a category and let τ be a topology on C. Then the results above provide a way of determining whether a presheaf $\mathcal{F}$ on C is a τ-sheaf, which is often useful in practice. Namely, if one can find a coverage σ that generates the topology τ, Proposition A.13 and Lemma A.10 show that if suffices to see that, for any family $\{f_i\colon x_i \to c\}$ in σ, the canonical morphism

$$\mathcal{F}(x) \to \lim\Big(\mathcal{F}(x) \to \prod_i \mathcal{F}(x_i) \rightrightarrows \prod_{i,j} \mathcal{F}(x_i \times_c x_j) \substack{\to\\\to\\\to} \dots\Big)$$

is an equivalence in $\mathrm{Psh}(\mathrm{Sm}_{/S})$. We use this kind of strategy for example in the proof of Theorem 1.15.

We will now study morphisms of sites:

Definition A.15. Let (C, τ) and $(\mathcal{D}, \tau')$ be sites with finite limits. A functor $\gamma\colon C \to \mathcal{D}$ is called a *morphism of sites*, if

i) the functor γ preserves finite limits and

ii) for every $c \in C$ and any family $\{x_i \to c\}$ generating a τ-covering sieve on c, the family $\{\gamma(x_i) \to \gamma(c)\}$ generates a τ'-covering sieve on $\gamma(c)$.

Let us recall the following definition:

Definition A.16. Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi. A *geometric morphism* from $\mathcal{X}$ to $\mathcal{Y}$ is given by a functor $f_*\colon \mathcal{X} \to \mathcal{Y}$ that has a left exact left adjoint $f^*\colon \mathcal{Y} \to \mathcal{X}$.

Proposition A.17. *Let $\gamma\colon (C, \tau) \to (\mathcal{D}, \tau')$ be a morphism of sites. Then the functor*

$$\mathfrak{c}^\gamma\colon\ \mathrm{Psh}(\mathcal{D}) \to \mathrm{Psh}(C),$$

given by precomposition with γ, restricts to a morphism

$$g_*\colon\ \mathrm{Sh}_{\tau'}(\mathcal{D}) \to \mathrm{Sh}_\tau(C),$$

which has a left adjoint g^* that preserves finite limits. So g_* is a geometric morphism of ∞-topoi.

Proof: First we will show that $\mathfrak{c}^\gamma$ restricts to a functor on categories of sheaves. Note that $\mathfrak{c}^\gamma$ has a left adjoint $\gamma_!\colon \mathrm{Psh}(C) \to \mathrm{Psh}(\mathcal{D})$ given by left Kan-extension. More explicitly, the adjoint $\gamma_!$ is the functor that is uniquely characterized by the fact that it preserves colimits and makes the diagram

$$\begin{array}{ccc} \mathrm{Psh}(C) & \longrightarrow & \mathrm{Psh}(\mathcal{D}) \\ \uparrow & & \uparrow \\ C & \xrightarrow{\ \gamma\ } & \mathcal{D} \end{array}$$

commute. Now let $\mathcal{F} \in \mathrm{Sh}_{\tau'}(\mathcal{D})$ and let $\{x_i \to c\}$ be a family generating a τ-covering sieve on $c \in C$. Consider the induced morphism

$$f\colon\ \coprod_i h(x_i) \to h(c)$$

in Psh(C). By Lemma A.10, we would like to show that the induced morphism

$$\mathfrak{c}^{\gamma}(\mathcal{F})(c) \to \mathrm{map}_{\mathrm{Psh}(C)}(|\check{C}_{\bullet}(f)|, \mathfrak{c}^{\gamma}(\mathcal{F}))$$

is an equivalence. By adjunction, this is equivalent to showing that the induced morphism

$$\mathcal{F}(\gamma(c)) \to \mathrm{map}_{\mathrm{Psh}(\mathcal{D})}(\gamma_!|\check{C}_{\bullet}(f)|, \mathcal{F}) \tag{1}$$

is an equivalence. But as $\gamma_!$ commutes with all colimits and agrees with γ when restricted to C and so in particular preserves pullbacks when restricted to C, it follows that $\gamma_!|\check{C}_{\bullet}(f)|$ can be identified with the Čech nerve of the morphism

$$\gamma_!(f)\colon \coprod_i h(\gamma(x_i)) \to h(\gamma(c))$$

induced by the family $\{\gamma(x_i) \to \gamma(c)\}$. As γ is a morphism of sites, this family generates a τ'-covering sieve. Therefore, the morphism (1) is an equivalence as $\mathcal{F}$ is a τ'-sheaf. So we have shown that $\mathfrak{c}^{\gamma}(\mathcal{F})$ is indeed a τ-sheaf and thus $\mathfrak{c}^{\gamma}$ restricts to a functor g_* of subcategories of sheaves, as claimed. Now we observe that g_* has a left adjoint g^*, given by the composite

$$g^*\colon \ \mathrm{Sh}_{\tau}(C) \hookrightarrow \mathrm{Psh}(C) \xrightarrow{\gamma_!} \mathrm{Psh}(\mathcal{D}) \xrightarrow{L_{(\mathcal{D},\tau')}} \mathrm{Sh}_{\tau'}(\mathcal{D}).$$

It remains to show that g^* preserves finite limits. But this follows since the first functor above preserves all limits and $L_{(\mathcal{D},\tau')} \circ \gamma_!$ preserves finite limits by [Lur09, Proposition 6.1.5.2]. □

Remark A.18. The above argument shows in particular that $\gamma_!\colon \mathrm{Psh}(C) \to \mathrm{Psh}(\mathcal{D})$ takes τ-covering sieves to τ'-covering sieves. It follows that $\gamma_!$ takes τ-local equivalences to τ'-local equivalences.

Example A.19. Let (C, τ) be a site with finite limits and let $c \in C$ be an object. Let $C_{/c}$ be the slice over c and let $F\colon C_{/c} \to C$ denote the forgetful functor. We equip $C_{/c}$ with a topology τ' by defining a family $\{x_i \to u\}$ of morphisms in $C_{/c}$ to generate a covering sieve if and only if the family $\{F(x_i) \to F(u)\}$ generates a covering sieve. It is easy to check that this indeed defines a topology on C. We have an adjunction

$$F\colon C_{/c} \rightleftarrows C : - \times c$$

and since covering families are stable under pullback, the functor $- \times c$ preserves covering families. Thus $- \times c$ is in fact a morphism of sites. So by Lemma A.17, we get an adjunction

$$g^*\colon \ \mathrm{Sh}_{\tau}(C) \rightleftarrows \mathrm{Sh}_{\tau'}(C_{/c}) : g_*,$$

where g^* preserves finite limits. But we also get an induced adjunction on the level of presheaf categories

$$\mathfrak{c}^F\colon \ \mathrm{Psh}(C) \rightleftarrows \mathrm{Psh}(C_{/c}) : \mathfrak{c}^{-\times c}$$

and it follows that $\mathfrak{c}^F \simeq (- \times c)_!$. Since F preserves pullbacks and covering families, the same argument as above shows that the further left adjoint $F_!$ of $\mathfrak{c}^F$ preserves local equivalences.

Therefore, the adjoint c^F takes τ-sheaves on C to τ'-sheaves on $C_{/c}$. It follows that g^* is given by the restriction of c^F and that it has a left adjoint itself, given by the composite

$$\mathrm{Sh}_{\tau'}(C_{/C}) \hookrightarrow \mathrm{Psh}(C_{/c}) \xrightarrow{F_!} \mathrm{Psh}(C) \xrightarrow{L_{(C,\tau)}} \mathrm{Sh}_\tau(C).$$

We will usually denote this left adjoint by $g_!$.

We will now quickly speak about hyperdescent. For this, we will start with recalling the notion of a hypercover:

Definition A.20.

i) Let $n \in \mathbb{N}$. We will write $\Delta^{\leq n} \subseteq \Delta$ for the full subcategory on the objects $[0], \ldots, [n]$. For C a presentable ∞-category, restriction along the inclusion $i\colon \Delta^{\leq n} \to \Delta$, denoted by

$$i^*\colon \mathrm{Fun}(\Delta^{\mathrm{op}}, C) \to \mathrm{Fun}((\Delta^{\leq n})^{\mathrm{op}}, C),$$

has a right adjoint i_* given by right Kan-extension. We will denote the composite $i_* \circ i^*$ by cosk_n.

ii) Let $\mathcal{X}$ be an ∞-topos. A simplicial object $U_\bullet\colon \Delta \to \mathcal{X}$ is called a *hypercovering* if the unit map induces an effective epimorphism

$$U_n \to \mathrm{cosk}_{n-1}(U_\bullet)_n$$

for all $n \in \mathbb{N}$.

Remark A.21. Let $f\colon X \to 1_{\mathcal{X}}$ be an effective epimorphism, where $1_{\mathcal{X}}$ is the terminal object of $\mathcal{X}$. Then the underlying groupoid of the Čech nerve (see [Lur09, Proposition 6.1.2.11]) of f

$$\check{C}(f)_\bullet\colon \Delta^{\mathrm{op}} \to \mathcal{X}$$

is the right Kan-extension of its restriction to $(\Delta^{\leq 0})^{\mathrm{op}}$. It follows that $\check{C}(f)_\bullet$ is a hypercovering.

A.22. Let $\mathcal{X}$ be an ∞-topos. We recall that a morphism $f\colon X \to Y$ in $\mathcal{X}$ is called ∞-connective if $\pi_k(f) = 0$ for all $k \in \mathbb{N}$ (see [Lur09, §6.5.1]). An object $X \in \mathcal{X}$ is called hypercomplete if and only if it is local with respect to ∞-connective morphisms. Furthermore, the full subcategory of hypercomplete objects, which we will denote by

$$\mathcal{X}^\wedge \subseteq \mathcal{X},$$

is an accessible localization of the ∞-topos $\mathcal{X}$ by [Lur09, Proposition 6.5.2.8]. Moreover, since ∞-connective morphisms are stable under pullback, it follows from [Lur09, Proposition 6.2.1.1] that the localization functor $L\colon \mathcal{X} \to \mathcal{X}^\wedge$ preserves finite limits. Thus $\mathcal{X}^\wedge$ itself is an ∞-topos and is called the *hypercompletion* of $\mathcal{X}$. See [Lur09, §6.5.2] for more details on the hypercompletion.

The relation between hypercompleteness and hypercoverings is the following:

Theorem A.23 ([Lur09, Corollary 6.5.3.13])**.** *Let $\mathcal{X}$ be an ∞-topos. For every $X \in \mathcal{X}$ and every hypercovering $U_\bullet$ in the ∞-topos $\mathcal{X}_{/X}$, consider the canonical morphism $c(U_\bullet)\colon |U_\bullet| \to X$. Let S be the collection of all such morphisms. Then an object $Y \in \mathcal{X}$ is hypercomplete if and only if it is S-local.*

A.24. Now let (C, τ) be a site. Informally speaking, the above theorem says that the ∞-topos $\mathrm{Sh}_\tau(C)^\wedge$ can be identified with the full subcategory of $\mathrm{Psh}(\mathrm{Sm}_{/S})$ of presheaves $\mathcal{F}$ that satisfy descent with respect to all covering families in τ and for all hypercoverings generated by τ. For that reason, we will refer to objects in $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_{/S})$ as *hypersheaves*.

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