The condensed homotopy type of a scheme

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with an appendix by Bogdan Zavyalov

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Abstract

We study a condensed version of the étale homotopy type of a scheme, which refines both the usual étale homotopy type of Friedlander–Artin–Mazur and the proétale fundamental group of Bhatt–Scholze. In the first part of this paper, we prove that this *condensed homotopy type* of schemes satisfies descent along integral morphisms and that the expected fiber sequences hold. We also provide explicit computations, for example for rings of continuous functions.

In the second part, we study the fundamental group of the condensed homotopy type in more detail. We show that, unexpectedly, the fundamental group of the condensed homotopy type of the affine line over the complex numbers, $\mathbf{A}^1_{\mathbf{C}}$, is nontrivial. However, we prove that the Noohi completion of the condensed fundamental group recovers the proétale fundamental group of Bhatt–Scholze. We also investigate a milder completion, the *quasi-separated quotient* of the condensed fundamental group. We show that this quotient already yields the expected answers and is, in some respects, even better behaved than the proétale fundamental group.

A key ingredient in many of our arguments is a description of the condensed homotopy type using the *Galois category* of a scheme, as introduced by Barwick–Glasman–Haine.

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1 Introduction

1.1 Motivation and overview

Let X be a locally topologically noetherian scheme. In their work on the proétale topology [10, §7], Bhatt and Scholze defined a refinement of the étale fundamental group called the *proétale* fundamental group $\pi_1^{\text{proét}}(X)$. The profinite completion of $\pi_1^{\text{proét}}(X)$ recovers the usual étale fundamental group; moreover, the proétale and étale fundamental groups coincide for normal schemes. While the étale fundamental group classifies local systems with values in profinite rings such as $\mathbf{Z}_\ell$, it generally does not classify $\mathbf{Q}_\ell$-local systems. The proétale fundamental group has the better feature that it classifies local systems in a more general class of topological rings, including $\mathbf{Q}_\ell$-local systems.

The étale fundamental group is the fundamental group of the *étale homotopy type*,[1] a proanima introduced by Artin–Mazur [7, §9] and Friedlander [23, §4]. The étale homotopy type classifies derived $\mathbf{Z}_\ell$-local systems, and has a number of interesting applications. For example, Friedlander's [21] and Sullivan's [74] proofs of the Adams Conjecture, Feng's proof [20] of Tate's 1966 conjecture on the Artin–Tate pairing [76], and applications to anabelian geometry [42; 68].

Motivated by the utility of the proétale fundamental group and the étale homotopy type, one desires a refinement of the Bhatt–Scholze proétale fundamental group to a 'homotopy type' that classifies derived $\mathbf{Q}_\ell$-local systems and refines the key properties of the étale homotopy type. The main goal of this article is to investigate such a refinement using the theory of condensed mathematics, introduced by Clausen-Scholze [70].

Condensed refinements of the étale homotopy type have already been defined or suggested in various places in the literature, by Bhatt-Scholze [10, Remark 4.2.9], Barwick-Glasman-Haine [8, 13.8.10], Hemo-Richarz-Scholbach [40, Appendix A], and Meffle [60]. In [40, Appendix A] and [8, §13.8] it is shown that the respective homotopy types indeed classify derived $\mathbf{Q}_l$-local systems. But beyond a few basic formal properties, little more was known about these refinements. Hence, the primary aim of this article is to undertake a thorough investigation of them.

The definitions given in [10], [40], and [60] are quite similar and proceed as follows. For a qcqs scheme X, pick a proétale hypercover $X_\bullet \to X$ by w-contractible schemes. Then for every $n \in \mathbf{N}$, $\pi_0(X_n)$ is a profinite set. Define the *condensed homotopy type* of X to be the colimit

$$\Pi_\infty^{\text{cond}}(X) = \operatorname*{colim}_{\mathbf{\Delta}^{\text{op}}} \pi_0(X_\bullet) \in \mathrm{Cond}(\mathbf{Ani}),$$

computed in the ∞-category $\mathrm{Cond}(\mathbf{Ani})$ of condensed anima.

This article consists of two parts. In the first part, we show that in many respects the condensed homotopy type behaves as one would expect from a refinement of the étale homotopy type. Among other results, we show that an analogue of the *fundamental fiber sequence* holds and that the condensed homotopy type satisfies *integral descent*; see Theorems 1.1 and 1.2 below. We also provide explicit computations of the condensed homotopy type, for example for rings of continuous functions $\mathrm{C}(T, \mathbf{C})$, where T is a compact Hausdorff space (see Theorem 1.3).

One of the main new tools that we use in many of our proofs relies on the work of Barwick–Glasman–Haine [8]. In loc. cit. the authors define a condensed category $\mathrm{Gal}(X)$, called the *Galois category* of a scheme. The aforementioned condensed refinement of the étale homotopy type in [8, 13.8.10] is the classifying space of this condensed category. We prove in Proposition 3.36 that this definition agrees with the others mentioned above, that is,

$$\Pi_\infty^{\text{cond}}(X) \simeq \mathrm{BGal}(X).$$

---

[1] Here, we really mean the étale fundamental group as defined in SGA3.

Since $\mathrm{Gal}(X)$ can be described somewhat explicitly, this is a useful tool in many proofs and calculations. For a more detailed account of the results we prove, see §1.2 below.

In the second part of this article, we investigate *the condensed fundamental group* of X. Every geometric point $\bar{x} \to X$ defines a point of the condensed anima $\Pi_\infty^{\mathrm{cond}}(X)$, giving rise to

$$\pi_n^{\mathrm{cond}}(X, \bar{x}) \coloneqq \pi_n(\Pi_\infty^{\mathrm{cond}}(X), \bar{x}).$$

Computing these groups is generally difficult, and the results can be wild and unexpected. For instance, we prove in Corollary 7.4 that the fundamental group of the affine line over the complex numbers is *nontrivial*:

$$\pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}}, \bar{x}) \neq 1.$$

While this departs from the classical situation, we show that the *Noohi completion* of $\pi_1^{\mathrm{cond}}(X, \bar{x})$ recovers the proétale fundamental group of Bhatt–Scholze; see Theorem 8.12. In fact, we prove that already the *quasi-separated quotient* $\pi_1^{\mathrm{cond,qs}}(X, \bar{x})$, a milder completion of $\pi_1^{\mathrm{cond}}(X, \bar{x})$, behaves computationally as expected (cf. Theorem 1.7). Studying $\pi_1^{\mathrm{cond,qs}}$ is another major theme of the second part of this article. It turns out that in some ways this quotient is even better behaved than $\pi_1^{\mathrm{proét}}$ (see, e.g., Remark 7.39). Using results from the first part, we establish the van Kampen and Künneth formulas for $\pi_1^{\mathrm{cond,qs}}$, allowing complete calculations for varieties over fields (cf. Theorems 1.9 and 1.10). For a more detailed account of the results we prove, see §1.2 below.

1.2 The condensed homotopy type

We now turn to explaining results that we prove in the first part of this paper in more detail. The first is a condensed version of the 'fundamental exact sequence' for the étale fundamental group.

1.1 Theorem (fundamental fiber sequence, Corollary 5.6)**.** *Let $f : X \to S$ be a morphism between qcqs schemes, and let $\bar{s} \to S$ be a geometric point of S. If* $\dim(S) = 0$*, then the naturally null sequence*

$$\Pi_\infty^{\mathrm{cond}}(X_{\bar{s}}) \longrightarrow \Pi_\infty^{\mathrm{cond}}(X) \longrightarrow \Pi_\infty^{\mathrm{cond}}(S)$$

is a fiber sequence in the ∞-category $\mathrm{Cond}(\mathbf{Ani})$*.*

The second is descent along hypercovers by integral surjections:

1.2 Theorem (integral hyperdescent, Corollary 6.16)**.** *The functor $X \mapsto X_{\mathrm{proét}}^{\mathrm{hyp}}$ sending a qcqs scheme X to its hypercomplete proétale ∞-topos satisfies integral hyperdescent. As a consequence, if $X_\bullet \twoheadrightarrow X$ is an integral hypercover, then the natural map of condensed anima*

$$\operatorname*{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \Pi_\infty^{\mathrm{cond}}(X_n) \to \Pi_\infty^{\mathrm{cond}}(X)$$

is an equivalence.

The description of the condensed homotopy type as $\mathrm{BGal}(X)$ is a crucial ingredient in our proof of the above theorem. Using this description, Theorem 1.2 follows rather quickly from the fact that, for an integral morphism of schemes $f : X \to Y$, the functor $\mathrm{Gal}(f)$ is a right fibration Proposition 6.9.

For the third, we give a complete computation of the condensed and étale homotopy types of rings of continuous functions to the complex numbers:

1.3 Theorem (Corollary 4.33)**.** *Let T be a compact Hausdorff space and consider the ring* $\mathrm{C}(T, \mathbf{C})$ *of continuous functions to the complex numbers. Then there is a natural equivalence of condensed anima*

$$\Pi_\infty^{\mathrm{cond}}(\mathrm{Spec}(\mathrm{C}(T, \mathbf{C}))) \simeq T\ .$$

(Here, the right-hand side denotes the condensed set represented by T.)

As a consequence, up to protruncation, the étale homotopy type of $\mathrm{Spec}(\mathrm{C}(T, \mathbf{C}))$ *is equivalent to the shape of the topological space T. In particular, if T admits a CW structure, then, up to protruncation, the étale homotopy type of* $\mathrm{Spec}(\mathrm{C}(T, \mathbf{C}))$ *recovers the underlying anima of T.*

1.4 Remark. The computation of the protruncated étale homotopy type of rings of continuous functions seems new. We also do not know of a direct computation that does not pass through the condensed homotopy type.

1.3 The condensed fundamental group

We now turn to our results about the condensed fundamental group. But first, let us remark that we also obtain a reasonably explicit description of the condensed set of connected components of $\Pi_\infty^{\mathrm{cond}}(X)$.

1.5 Theorem (Theorem 4.17 and Corollary 4.19)**.** *Let X be a qcqs scheme. Then, for any extremally disconnected profinite set S, we have*

$$\pi_0^{\mathrm{cond}}(X)(S) = \mathrm{Map}_{\mathrm{qc}}(S, |X|)/\sim\ ,$$

where $\sim$ is the equivalence relation generated by pointwise specializations.

In particular, if X has finitely many irreducible components, then $\pi_0^{\mathrm{cond}}(X)$ coincides with the usual profinite set $\pi_0(X)$ of connected components of X.

1.6 Remark (see Example 4.23)**.** Let R be a ring with the property that $|\mathrm{Spec}(R)|$ is homeomorphic to the underlying spectral space of Huber's adic unit disk over $\mathbf{Q}_p$. Then the condensed set $\pi_0^{\mathrm{cond}}(\mathrm{Spec}(R))$ coincides with the *separated quotient* of the space $|\mathrm{Spec}(R)|$. This is a compact Hausdorff space, and moreover, it coincides with the Berkovich unit disk, i.e.,

$$\pi_0^{\mathrm{cond}}(\mathrm{Spec}(R)) \simeq |\mathbf{D}^{1,\mathrm{Berk}}_{\mathbf{Q}_p}|\ .$$

While this example feels rather contrived in the realm of schemes, in a follow-up article we plan to study a similarly defined condensed homotopy type for rigid spaces.

We now turn to our results about the condensed fundamental group. As stated before, the condensed group $\pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}})$ is nontrivial. Our first result is that a quotient more mild than Noohi completion forces $\pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}})$ to become trivial. Specifically, Clausen and Scholze introduced a localization $A \mapsto A^{\mathrm{qs}}$ of the category of condensed sets called the *quasiseparated quotient* [69, Lecture VI], and we show:

1.7 Theorem (Theorem 7.17)**.** *Let X be a topologically noetherian scheme that is geometrically unibranch and let $\bar{x} \to X$ be a geometric point. Then there is a natural isomorphism*

$$\pi_1^{\mathrm{cond,qs}}(X, \bar{x}) \xrightarrow{\sim} \pi_1^{\mathrm{\acute{e}t}}(X, \bar{x})\ .$$

As a consequence of Theorems 1.1 and 1.5, we deduce a fundamental exact sequence for the quasiseparated quotient of the condensed fundamental group:

1.8 Theorem (fundamental exact sequence, Corollary 7.16)**.** *Let k be a field with separable closure $\bar{k}$, let X be a qcqs k-scheme, and fix a geometric point $\bar{x} \to X_{\bar{k}}$. If X is geometrically connected and $X_{\bar{k}}$ has finitely many irreducible components, then the sequence*

$$1 \longrightarrow \pi_1^{\mathrm{cond,qs}}(X_{\bar{k}}, \bar{x}) \longrightarrow \pi_1^{\mathrm{cond,qs}}(X, \bar{x}) \longrightarrow \mathrm{Gal}_k \longrightarrow 1 \,.$$

is exact.

Theorem 1.7 can be used, together with the integral descent (Theorem 1.2), to show that for many non-normal schemes, the quasiseparated quotient of the condensed fundamental group still admits a description in terms of the étale fundamental group. Moreover, surprisingly, is a topological (Hausdorff) group rather than some more complicated condensed group.

1.9 Theorem (van Kampen formula for $\pi_1^{\mathrm{cond,qs}}$, special case of Theorem 7.35)**.** *Let X be a Nagata qcqs scheme and let $X^\nu = \coprod_i X_i^\nu$ be its normalization decomposed into connected components. After choosing base points and étale paths, one has that*

$$\pi_1^{\mathrm{cond,qs}}(X, \bar{x}) \simeq \underline{\big(*_i^{\mathrm{top}}\ \pi_1^{\text{ét}}(X_i^\nu, \bar{x}_i) *^{\mathrm{top}} \mathbf{Z}^{*r} \big)/H'} \,,$$

where $\mathbf{Z}^{*r}$ is a free (discrete) group of finite rank, $*^{\mathrm{top}}$ denotes the free topological product and H' is an explicit closed normal subgroup.

Using the van Kampen and the Künneth formulas for the étale fundamental group, we prove:

1.10 Theorem (Corollary 7.37)**.** *Let k be a separably closed field and let X and Y be schemes of finite type over k. If Y is proper or* $\mathrm{char}(k) = 0$*, then the natural homomorphism of condensed groups*

$$\pi_1^{\mathrm{cond,qs}}(X \times_k Y, (\bar{x}, \bar{y})) \to \pi_1^{\mathrm{cond,qs}}(X, \bar{x}) \times \pi_1^{\mathrm{cond,qs}}(Y, \bar{y})$$

is an isomorphism.

1.4 Related work

As mentioned earlier, the first definitions of the condensed homotopy type were given by Barwick–Glasman–Haine via exodromy [8, 13.8.10], by Bhatt–Scholze [10, Remark 4.2.9] and by Hemo–Richarz–Scholbach [40, Appendix A]. We expand the definitions given there by the perspective of relative shape and, more importantly, show that all of these are equivalent. Another approach to the condensed homotopy type that mostly uses (simplicial) topological spaces rather than condensed mathematics (along the lines of Artin and Mazur's work) appeared in [60].

Some results and definitions in this article constitute a part of doctoral theses of the forth [56] and sixth [81] named authors.

1.5 Linear overview

In §2, we recall some preliminaries on condensed anima, pro-objects, condensed ∞-categories, and proétale sheaves. In §3, we recall the various definitions of the condensed homotopy type and prove that they are all equivalent. We also compute the condensed homotopy type of henselian local rings (Corollary 3.44). In § 4, we describe the connected components of the condensed homotopy type. Among other things, we show that if X is a qcqs scheme with finitely many irreducible components, then $\pi_0^{\mathrm{cond}}(X)$ is simply the profinite set $\pi_0(X)$ of connected components of

X (Corollary 4.19). As an application of our explicit description of $\pi_0^{\mathrm{cond}}(X)$, we also we compute the condensed homotopy type of rings of continuous functions (Theorem 1.3)

In § 5, we prove the fundamental fiber sequence Theorem 1.1. We also prove an analogue of a result of Friedlander relating the condensed homotopy type of the geometric fiber of a smooth proper morphism to the fiber of the induced map on condensed homotopy types (Theorem 5.12). In § 6, we prove that the condensed homotopy type satisfies integral hyperdescent (Theorem 1.2).

We then turn our attention to the condensed fundamental group. In § 7, we study the quasiseparated quotient of the condensed fundamental group. We begin by showing that $\pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}}) \neq 1$. Then we prove Theorems 1.7 and 1.10, along with the van Kampen theorem and fundamental exact sequence for the quasiseparated quotient. In § 8, we prove that the Noohi completion of the condensed fundamental group recovers the proétale fundamental group.

We have three appendices. Appendix A, by Bogdan Zavyalov, is on the structure of rings of continuous functions and the relationship between these rings and Čech–Stone compactification. We need these results for the computation of the condensed homotopy type of rings of continuous functions, however were not able to find any sources that contained all of the results we needed.

It is well-known that there is an abstract isomorphism between the absolute Galois group of the function field $\mathbf{C}(t)$ and the free profinite group on the set $\mathbf{C}$. See, for example [19; 38; 46]. It seems to be folklore that this isomorphism can be chosen to be compatible with decomposition groups; this is crucial for our proof that $\pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}}) \neq 1$. Since we could not find this proven in the literature, and there are some subtleties involved, we have included a proof in Appendix B. In Appendix C, we prove a version of Quillen's Theorem B for profinite categories, which is a crucial ingredient in § 5.2.

1.6 Notational conventions

We use the following standard notation.

(1) We write $\mathbf{Cat}_\infty$ for the large ∞-category of small ∞-categories, and write $\mathbf{Ani} \subset \mathbf{Cat}_\infty$ for the full subcategory spanned by the anima (also called ∞-groupoids or spaces).

(2) Given a small ∞-category $\mathcal{C}$, we write $\mathrm{PSh}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Ani})$ for the ∞-category of presheaves of anima on $\mathcal{C}$.

(3) Given an ∞-topos $\mathcal{X}$, we write $\mathcal{X}^{\mathrm{hyp}} \subset \mathcal{X}$ for the full subcategory spanned by the hypercomplete objects. The inclusion is accessible and admits a left exact accessible left adjoint, so that $\mathcal{X}^{\mathrm{hyp}}$ is also an ∞-topos, called the *hypercompletion* of $\mathcal{X}$.

(4) Given an ∞-site $(\mathcal{C}, \tau)$, we write $\mathrm{Sh}_\tau(\mathcal{C})$ for the ∞-topos of sheaves of anima on $\mathcal{C}$ with respect to τ. We write $\mathrm{Sh}_\tau^{\mathrm{hyp}}(\mathcal{C}) := \mathrm{Sh}_\tau(\mathcal{C})^{\mathrm{hyp}}$. The ∞-topos $\mathrm{Sh}_\tau^{\mathrm{hyp}}(\mathcal{C})$ can also be identified as the full subcategory of $\mathrm{Sh}_\tau(\mathcal{C})$ spanned by those sheaves that also satisfy descent for *hypercovers*. If the topology τ is clear from the context, we may omit it from the notation.

(5) Given a scheme X, we write $\mathrm{Ét}_X$ and $\mathrm{ProÉt}_X$ for its *étale* and *proétale site*, respectively. Moreover, we write $X_{\mathrm{ét}} := \mathrm{Sh}(\mathrm{Ét}_X)$ and $X_{\mathrm{proét}} := \mathrm{Sh}(\mathrm{ProÉt}_X)$ for the ∞-topoi of étale and proétale sheaves of anima on X, respectively.

(6) For an integer $n \geq 0$, we write $[n]$ for the poset $\{0 < \ldots < n\}$.

(7) For each integer $n \geq 0$, we write $\mathbf{\Delta}_{\leq n} \subset \mathbf{\Delta}$ for the full subcategory spanned by $[0]$, $[1]$, …, $[n]$.

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2 Preliminaries

For later use, let us record a few definitions and observations on condensed anima (§ 2.1), pro-anima and their relation to condensed anima (§ 2.2), condensed ∞-categories (§ 2.3), shape theory (§ 2.4), and proétale sheaves and w-contractible objects (§ 2.5).

2.1 Recollection on condensed anima

All of the material contained in this subsection is gathered from [9] and [70].

2.1 Notation. We write **Top** for the category of topological spaces, and $\mathbf{Comp} \subset \mathbf{Top}$ for the full subcategory spanned by the compact Hausdorff spaces. We write $\beta\colon \mathbf{Top} \to \mathbf{Comp}$ for the Čech–Stone compactification functor, i.e., the left adjoint to the inclusion. By Stone duality, the category $\mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$ of profinite sets embeds fully faithfully into **Comp** with image the full subcategory spanned by the totally disconnected compact Hausdorff spaces. We write

$$\mathbf{Extr} \subset \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$$

for the full subcategory spanned by the *extremally disconnected* profinite sets. By a theorem of Gleason [29], the projective objects of the category **Comp** are exactly the extremally disconnected profinite sets. Moreover, a profinite set is extremally disconnected if and only if it is a retract of the Čech–Stone compactification of a set equipped with the discrete topology.

2.2 Recollection (condensed anima)**.** Give the category **Comp** of compact Hausdorff spaces the Grothendieck topology where the covering families are generated by finite jointly surjective families. For each compact Hausdorff space T, let T^{δ} denote the underlying set of T equipped with the discrete topology. By the universal property of Čech–Stone compactification the ‘identity’ map $T^\delta \to T$ extends to a surjection $\beta(T^\delta) \twoheadrightarrow T$. In particular, every compact Hausdorff space admits a surjection from an extremally disconnected profinite set. Hence the subcategories

$$\mathbf{Extr} \subset \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}}) \subset \mathbf{Comp}$$

are bases for the topology of finite jointly surjective families. By [3, Corollary A.7], the restriction functors define equivalences hypercomplete of ∞-topoi

$$\mathrm{Sh}^{\mathrm{hyp}}(\mathbf{Comp}) \xrightarrow{\sim} \mathrm{Sh}^{\mathrm{hyp}}(\mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})) \xrightarrow{\sim} \mathrm{Sh}^{\mathrm{hyp}}(\mathbf{Extr}) . \tag{2.3}$$

The ∞-topos $\mathrm{Cond}(\mathbf{Ani})$ of *condensed anima* is any of the equivalent ∞-topoi (2.3).

Since every surjection $T' \twoheadrightarrow T$ of profinite sets with T extremally disconnected admits a section, a presheaf F on $\mathbf{Extr}$ is a hypersheaf if and only if F carries finite disjoint unions to finite products. That is,

$$\mathrm{Sh}^{\mathrm{hyp}}(\mathbf{Extr}) \simeq \mathrm{Fun}^{\times}(\mathbf{Extr}^{\mathrm{op}}, \mathbf{Ani}) .$$

From this description it follows that sifted colimits in $\mathrm{Cond}(\mathbf{Ani})$ can be computed in the presheaf category $\mathrm{Fun}(\mathbf{Extr}^{\mathrm{op}}, \mathbf{Ani})$.

2.4 Remark. Since the category **Comp** of compact Hausdorff spaces is not a small category, there are some set-theoretic issues in the above discussion. We explain how to deal with these issues in Remark 2.30.

Given the final description of condensed anima, we make the following convenient general definition.

2.5 Definition (condensed objects)**.** Let $\mathcal{C}$ be an ∞-category with finite products. The ∞-category of *condensed objects* of $\mathcal{C}$ is the ∞-category

$$\mathrm{Cond}(\mathcal{C}) \coloneqq \mathrm{Fun}^{\times}(\mathbf{Extr}^{\mathrm{op}}, \mathcal{C})$$

of finite product-preserving presheaves $\mathbf{Extr}^{\mathrm{op}} \to \mathcal{C}$. If $\mathcal{D}$ is another ∞-category with finite products and $F \colon \mathcal{C} \to \mathcal{D}$ is a finite product-preserving functor, we write

$$F^{\mathrm{cond}} \colon \mathrm{Cond}(\mathcal{C}) \to \mathrm{Cond}(\mathcal{D})$$

for the functor given by post-composition with F.

2.6. Observe that if $F \colon \mathcal{C} \to \mathcal{D}$ admits a right adjoint G, then G^{cond} is right adjoint to F^{cond}.

2.7 Recollection (homotopy groups of condensed anima)**.** The functor $\pi_0 \colon \mathbf{Ani} \to \mathbf{Set}$ preserves finite products. Moreover, for each integer $n \geq 1$, the functor $\pi_n \colon \mathbf{Ani}_* \to \mathbf{Grp}$ preserves finite products. There is a canonical identification

$$\mathrm{Cond}(\mathbf{Ani})_* = \mathrm{Cond}(\mathbf{Ani}_*)$$

between pointed objects of condensed anima and condensed objects of pointed anima. We simply write $\pi_0 \colon \mathrm{Cond}(\mathbf{Ani}) \to \mathrm{Cond}(\mathbf{Set})$ for π_0^{cond} and $\pi_n \colon \mathrm{Cond}(\mathbf{Ani})_* \to \mathrm{Cond}(\mathbf{Grp})$ for

$$\mathrm{Cond}(\mathbf{Ani})_* = \mathrm{Cond}(\mathbf{Ani}_*) \xrightarrow{\pi_n^{\mathrm{cond}}} \mathrm{Cond}(\mathbf{Grp}) .$$

Explicitly, given a condensed anima A, the condensed set $\pi_0(A) \colon \mathbf{Extr}^{\mathrm{op}} \to \mathbf{Set}$ is given by

$$\pi_0(A)(S) \coloneqq \pi_0(A(S)) .$$

Similarly, given a global section $a \colon * \to A$, the condensed group $\pi_n(A, a)$ is given by

$$\pi_n(A, a)(S) \coloneqq \pi_n(A(S), a) .$$

2.8 Recollection [9, Construction 2.2.12]**.** Write

$$\mathrm{ev}_* : \ \mathrm{Cond}(\mathbf{Ani}) \to \mathbf{Ani}$$

for the global sections functor, given by $A \mapsto A(*)$. The functor ev_* admits a left adjoint, that we denote by

$$(-)^{\mathrm{disc}} : \ \mathbf{Ani} \to \mathrm{Cond}(\mathbf{Ani})$$

Furthermore $(-)^{\mathrm{disc}}$ is fully faithful. We call the image of $(-)^{\mathrm{disc}}$ the *discrete* condensed anima.

2.9 Recollection (the restricted Yoneda embedding)**.** The restricted Yoneda embedding defines a functor

$$\mathbf{Top} \to \mathrm{Fun}^{\times}(\mathbf{Extr}^{\mathrm{op}}, \mathbf{Ani}) = \mathrm{Cond}(\mathbf{Ani}) \ , \ T \mapsto \underline{T}$$

given by

$$T \mapsto [S \mapsto \mathrm{Map}_{\mathbf{Top}}(S, T)] \ .$$

Note that this functor factors through $\mathrm{Cond}(\mathbf{Set}) \subset \mathrm{Cond}(\mathbf{Ani})$.[2] Also recall that this functor is fully faithful when restricted to the full subcategory of **Top** spanned by the compactly generated topological spaces [70, Proposition 1.7]. Since it rarely leads to confusion, we often omit the underline and simply write T for $\underline{T}$.

2.2 Pro-objects and completions

We now turn to some recollections about proanima and their relation to condensed anima.

2.10 Recollection (π-finite and truncated anima)**.** Let A be an anima.

(1) We say that A is *truncated* if there exists an integer $n \geq 0$ such that for all $a \in A$ and $k \geq n$, we have $\pi_k(A, a) = 0$.

(2) We say that A is *π-finite* if A is truncated, $\pi_0(A)$ is finite, and for all $a \in A$ and $k > 0$, the group $\pi_k(A, a)$ is finite.

(3) We write $\mathbf{Ani}_\pi \subset \mathbf{Ani}_{<\infty} \subset \mathbf{Ani}$ for the full subcategories of **Ani** spanned by the π-finite and truncated anima, respectively.

2.11 Recollection (on various completions)**.**

(1) Since $\mathrm{Cond}(\mathbf{Ani})$ admits cofiltered limits, the inclusions

$$\mathbf{Ani}_\pi \subset \mathbf{Ani}_{<\infty} \subset \mathrm{Cond}(\mathbf{Ani})$$

extend to cofiltered-limit-preserving functors

$$\mathrm{Pro}(\mathbf{Ani}_\pi) \hookrightarrow \mathrm{Pro}(\mathbf{Ani}_{<\infty}) \to \mathrm{Cond}(\mathbf{Ani}) \ .$$

Here, the functor $\mathrm{Pro}(\mathbf{Ani}_{<\infty}) \to \mathrm{Cond}(\mathbf{Ani})$ is *not* fully faithful. However, by [9, Example 3.3.10; 34, Proposition 0.1], its restriction to $\mathrm{Pro}(\mathbf{Ani}_\pi)$ is fully faithful.

---

[2] However, note that if T is not T_1, then the the sheaf $\mathrm{Map}_{\mathbf{Top}}(-, T)$ is not generally *accessible* [70, Warning 2.14 & Proposition 2.15]. So, depending on which way you deal with set-theoretic issues, it is not a condensed set, cf. Remark 2.30. However, in this paper, we only apply this functor to T_1 topological spaces anyways.

(2) The above chain of functors $\mathrm{Pro}(\mathbf{Ani}_{\pi}) \hookrightarrow \mathrm{Pro}(\mathbf{Ani}_{<\infty}) \to \mathrm{Cond}(\mathbf{Ani})$ admits left adjoints

$$\begin{tikzcd}
\mathrm{Cond}(\mathbf{Ani}) \arrow[r, "(-)^{\wedge}_{\mathrm{disc}}"'] \arrow[rr, bend left, "(-)^{\wedge}_{\pi}"] & \mathrm{Pro}(\mathbf{Ani}_{<\infty}) \arrow[r, "(-)^{\wedge}_{\pi}"'] & \mathrm{Pro}(\mathbf{Ani}_{\pi})
\end{tikzcd}$$

that we call the *prodiscretization*, resp., *profinite completion* functors.

(3) Similarly, the inclusions $\mathbf{Set}_{\mathrm{fin}} \subset \mathrm{Cond}(\mathbf{Set})$ and $\mathbf{Grp}_{\mathrm{fin}} \subset \mathrm{Cond}(\mathbf{Grp})$ induce inclusions $\mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}}) \subset \mathrm{Cond}(\mathbf{Set})$ and $\mathrm{Pro}(\mathbf{Grp}_{\mathrm{fin}}) \subset \mathrm{Cond}(\mathbf{Grp})$ that admit left adjoints

$$\mathrm{Cond}(\mathbf{Set}) \to \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}}) \qquad \text{and} \qquad (-)^{\wedge} \colon \mathrm{Cond}(\mathbf{Grp}) \to \mathrm{Pro}(\mathbf{Grp}_{\mathrm{fin}})$$

that we refer to as *profinite completion* functors.

We now explain the effect of profintie completion of condensed anima on π_0 and π_1.

2.12 Lemma (completions & π_0/π_1)**.** *Let A be a condensed anima and $a \colon * \to A$ a point.*

(1) *The map $\pi_0(A) \to \pi_0(A^{\wedge}_{\pi})$ induced by the unit map $A \to A^{\wedge}_{\pi}$ exhibits $\pi_0(A^{\wedge}_{\pi})$ as the profinite completion of $\pi_0(A)$.*

(2) *If $\pi_0(A) \in \mathrm{Cond}(\mathbf{Set})$ is discrete, then the unit map $A \to A^{\wedge}_{\pi}$ induces an isomorphism of profinite groups*

$$\pi_1(A, a)^{\wedge} \xrightarrow{\sim} \pi_1(A^{\wedge}_{\pi}, a)\,.$$

Proof. For (1), note that since the square of inclusions

$$\begin{tikzcd}
\mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}}) \arrow[r, hook] \arrow[d, hook] & \mathrm{Cond}(\mathbf{Set}) \arrow[d, hook] \\
\mathrm{Pro}(\mathbf{Ani}_{\pi}) \arrow[r, hook] & \mathrm{Cond}(\mathbf{Ani})
\end{tikzcd}$$

commutes, so does the induced square

$$\begin{tikzcd}
\mathrm{Cond}(\mathbf{Ani}) \arrow[r, "(-)^{\wedge}_{\pi}"] \arrow[d, "\pi_0"'] & \mathrm{Pro}(\mathbf{Ani}_{\pi}) \arrow[d, "\pi_0"] \\
\mathrm{Cond}(\mathbf{Set}) \arrow[r] & \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})
\end{tikzcd}$$

of left adjoints.

For (2), since $\pi_0(A)$ is a set, we may assume that $\pi_0(A) = *$. It suffices to show that, for any finite group G, precomposition induces a bijection

$$\mathrm{Map}_{\mathrm{Cond}(\mathbf{Grp})}(\pi_1(A, a), G) \xleftarrow{\sim} \mathrm{Map}_{\mathrm{Cond}(\mathbf{Grp})}(\pi_1(A^{\wedge}_{\pi}, a), G) = \mathrm{Map}_{\mathrm{Pro}(\mathbf{Grp}_{\mathrm{fin}})}(\pi_1(A^{\wedge}_{\pi}, a), G)\,.$$

To see this, note that we have a commutative square

$$\begin{tikzcd}
\pi_0\mathrm{Map}_{\mathrm{Pro}(\mathbf{Ani}_{\pi})_*}(A^{\wedge}_{\pi}, \mathrm{B}G) \arrow[r, "\pi_1", "\sim"'] \arrow[d] & \mathrm{Map}_{\mathrm{Pro}(\mathbf{Grp}_{\mathrm{fin}})}(\pi_1(A^{\wedge}_{\pi}, a), G) \arrow[d] \\
\pi_0\mathrm{Map}_{\mathrm{Cond}(\mathbf{Ani})_*}(A, \mathrm{B}G) \arrow[r, "\sim", "\pi_1"'] & \mathrm{Map}_{\mathrm{Cond}(\mathbf{Grp})}(\pi_1(A, a), G),
\end{tikzcd}$$

where the vertical maps are those induced by the unit transformation $A \to A^{\wedge}_{\pi}$. Since $\pi_0(A) = *$, by the equivalence of 1-truncated, pointed connected objects and group objects [HTT, Theorem 7.2.2.12], the horizontal maps are bijections. It thus suffices to see that the map

$$\mathrm{Map}_{\mathrm{Cond}(\mathbf{Ani})_*}(A^{\wedge}_{\pi}, \mathrm{B}G) \to \mathrm{Map}_{\mathrm{Cond}(\mathbf{Ani})_*}(A, \mathrm{B}G)$$

induces a bijection on π_0. But since G is finite and $\mathrm{Pro}(\mathbf{Ani}_{\pi})_* \hookrightarrow \mathrm{Cond}(\mathbf{Ani})_*$ is fully faithful, by adjunction it is even an equivalence. □

2.13 Remark. One cannot drop the assumption that $\pi_0(A)$ is discrete in Lemma 2.12 (2). Indeed, let A be the condensed *set* represented by the topological circle S^1. Then for any $x \in \mathrm{S}^1$, we have

$$\pi_1(A, x) = * \quad \text{but} \quad \pi_1(A^{\wedge}_{\pi}, x) = \hat{\mathbf{Z}}\,.$$

2.3 Condensed ∞-categories

We now recall some background on internal higher category theory and condensed ∞-categories. The main point is that it is often useful to use the fact that the ∞-category of condensed ∞-categories is equivalent to the ∞-category of categories internal to condensed anima. We refer the reader to [57, §3; 59, §2] for more background about internal higher category theory.

2.14 Definition. Let $\mathcal{B}$ be an ∞-category with finite limits. A *category internal to* $\mathcal{B}$ is a simplicial object $F \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{B}$ satisfying the following conditions.

(1) *Segal condition:* For each integer $n \geq 2$, the natural map

$$F([n]) \to F(\{0<1\}) \underset{F(\{1\})}{\times} F(\{1<2\}) \underset{F(\{2\})}{\times} \cdots \underset{F(\{n-1\})}{\times} F(\{n-1<n\})$$

is an equivalence in $\mathcal{B}$.

(2) *Univalence axiom:* The natural square

$$\begin{tikzcd} F([0]) \arrow[r, "\Delta"] \arrow[d] & F([0]) \times F([0]) \arrow[d] \\ F([3]) \arrow[r] & F(\{0<2\}) \times F(\{1<3\}) \end{tikzcd}$$

is a pullback square in $\mathcal{B}$. Here, the left vertical map is given by restriction along the unique map $[3] \to [0]$, the right vertical map is the product of the maps given by restriction along the unique maps $\{0<2\} \to [0]$ and $\{1<3\} \to [0]$, and the bottom horizontal map is induced by restriction along the inclusions $\{0<2\} \hookrightarrow [3]$ and $\{1<3\} \hookrightarrow [3]$.

We write

$$\mathrm{Cat}(\mathcal{B}) \subset \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{B})$$

for the full subcategory spanned by the categories internal to $\mathcal{B}$.

2.15 Remark. Elsewhere in the literature, internal categories are also called *complete Segal objects*.

2.16. Joyal and Tierney [48] showed that the nerve construction defines an equivalence

$$\begin{aligned} \mathrm{N}\colon \mathbf{Cat}_\infty &\xrightarrow{\sim} \mathrm{Cat}(\mathbf{Ani}) \\ C &\mapsto [[n] \mapsto \mathrm{Map}_{\mathbf{Cat}_\infty}([n], C)] \end{aligned}$$

from the ∞-category of ∞-categories to the ∞-category of categories internal to anima. See [39] for a modern, model-independent proof of this fact.

2.17. The main example that we care about in this paper is the case where $\mathcal{B} = \mathrm{Cond}(\mathbf{Ani})$. Since the Segal conditions and the sheaf condition are both limit conditions, the canonical equivalence

$$\mathrm{Fun}(\mathbf{Extr}^{\mathrm{op}}, \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Ani})) \simeq \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Fun}(\mathbf{Extr}^{\mathrm{op}}, \mathbf{Ani}))$$

restricts to an equivalence

$$\mathrm{Cond}(\mathbf{Cat}_\infty) \simeq \mathrm{Cat}(\mathrm{Cond}(\mathbf{Ani}))\ .$$

Therefore, we often implicitly identify $\mathrm{Cond}(\mathbf{Cat}_\infty)$ with $\mathrm{Cat}(\mathrm{Cond}(\mathbf{Ani}))$.

We now turn to some specific features of $\mathrm{Cond}(\mathbf{Cat}_\infty)$.

2.18 Definition (continuous functors)**.** The category of condensed ∞-categories is cartesian closed, see [57, Proposition 3.2.11]. For condensed ∞-categories $\mathcal{C}$ and $\mathcal{D}$, we denote the internal Hom by

$$\mathrm{Fun}^{\mathrm{cond}}(\mathcal{C}, \mathcal{D})\ .$$

Similarly, we write

$$\mathrm{Fun}^{\mathrm{cts}}(\mathcal{C}, \mathcal{D}) \coloneqq \mathrm{Fun}^{\mathrm{cond}}(\mathcal{C}, \mathcal{D})(*)$$

for the ∞-category of *continuous functors* $\mathcal{C} \to \mathcal{D}$.

2.19. Observe that the functor $(\mathcal{C}, \mathcal{D}) \mapsto \mathrm{Fun}^{\mathrm{cts}}(\mathcal{C}, \mathcal{D})$ is characterized by the existence of natural equivalences

$$\mathrm{Map}_{\mathbf{Cat}_\infty}(\mathcal{A}, \mathrm{Fun}^{\mathrm{cts}}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Map}_{\mathrm{Cond}(\mathbf{Cat}_\infty)}(\mathcal{A} \times \mathcal{C}, \mathcal{D})$$

for each ∞-category $\mathcal{A}$.

2.20. Explicitly, $\mathrm{Fun}^{\mathrm{cts}}(\mathcal{C}, \mathcal{D})$ is given by the end

$$\mathrm{Fun}^{\mathrm{cts}}(\mathcal{C}, \mathcal{D}) \simeq \int_{S \in \mathbf{Extr}^{\mathrm{op}}} \mathrm{Fun}(\mathcal{C}(S), \mathcal{D}(S))\ ,$$

see, for example, [28, Proposition 2.3]. In particular, the objects in this ∞-category are precisely natural transformations $\mathcal{C}(-) \to \mathcal{D}(-)$ of functors $\mathbf{Extr}^{\mathrm{op}} \to \mathbf{Cat}_\infty$.

Many of the condensed ∞-categories we are interested come from pro-objects:

2.21 Observation. By taking internal categories on each side, the right adjoint fully faithful embedding $\mathrm{Pro}(\mathbf{Ani}_\pi) \to \mathrm{Cond}(\mathbf{Ani})$ of Recollection 2.11 induces a fully faithful right adjoint functor

$$\iota\colon \mathrm{Cat}(\mathrm{Pro}(\mathbf{Ani}_\pi)) \to \mathrm{Cond}(\mathbf{Cat}_\infty)\ .$$

Many of the examples of condensed ∞-categories that we care about are in the image of this embedding.

For condensed ∞-categories in the image of ι, we can describe their value at Čech–Stone compactifations explicitly:

2.22 Proposition. *Consider* $\mathcal{C} \in \mathrm{Cat}(\mathrm{Pro}(\mathbf{Ani}_\pi))$ *as a condensed* ∞*-category via* ι *and let* M *be a set. Then the functor*

$$\mathrm{Fun}^{\mathrm{cts}}(\beta(M), \mathcal{C}) \to \prod_{m \in M} \mathcal{C}(\{m\})$$

induced by the inclusions $\{m\} \hookrightarrow \beta(M)$ *is an equivalence of* ∞*-categories.*

Proof. It suffices to check that this functor becomes an equivalence after applying $\mathrm{Map}_{\mathbf{Cat}_\infty}([n], -)$ for every n. Since we have a natural chain of equivalences

$$\begin{aligned} \mathrm{Map}_{\mathbf{Cat}_\infty}([n], \mathrm{Fun}^{\mathrm{cts}}(\beta(M), \mathcal{C})) &\simeq \mathrm{Map}_{\mathrm{Cond}(\mathbf{Cat}_\infty)}(\beta(M) \times [n], \mathcal{C}) \\ &\simeq \mathrm{Map}_{\mathrm{Cond}(\mathbf{Cat}_\infty)}(\beta(M), \mathrm{ev}_{[n]}(\mathcal{C})), \end{aligned}$$

it suffices to show that the natural map

$$\mathrm{Map}_{\mathrm{Cond}(\mathbf{Cat}_\infty)}(\beta(M), \mathrm{ev}_{[n]}(\mathcal{C})) \to \prod_{m \in M} \mathrm{ev}_{[n]}(\mathcal{C})(\{m\})$$

is an equivalence. Since $\mathrm{ev}_{[n]}(\mathcal{C})$ is a profinite anima by assumption and both sides are clearly compatible with limits, we may assume that $\mathrm{ev}_{[n]}(\mathcal{C}) = A$ is a π-finite anima.

By [SAG, Lemma E.1.6.5], there exists a Kan complex $A_\bullet$ with values in finite sets such that $|A_\bullet| \simeq A$. Since $\beta(M)$ is a compact projective object in $\mathrm{Cond}(\mathbf{Ani})$, it follows that the natural map

$$|\mathrm{Map}_{\mathrm{Cond}(\mathbf{Ani})}(\beta(M), A_\bullet)| \to \mathrm{Map}_{\mathrm{Cond}(\mathbf{Ani})}(\beta(M), |A_\bullet|)$$

is an equivalence. Since every A_n is finite, it follows that $\mathrm{Map}_{\mathrm{Cond}(\mathbf{Ani})}(\beta(M), A_\bullet) \simeq \prod_M A_\bullet$ is an infinite product of Kan complexes. Since geometric realizations of Kan complexes commute with arbitrary products,[3] the natural map

$$\mathrm{Map}_{\mathrm{Cond}(\mathbf{Ani})}(\beta(M), A) \simeq |\mathrm{Map}_{\mathrm{Cond}(\mathbf{Ani})}(\beta(M), A_\bullet)| \longrightarrow \prod_M |A_\bullet| \simeq \prod_M A$$

is an equivalence. □

2.4 Recollection on shape theory

2.23 Recollection. For every ∞-topos $\mathcal{X}$, there exists a unique geometric morphism $g \colon \mathcal{X} \to \mathbf{Ani}$ and the pullback functor g^* admits a pro-left adjoint $g_\sharp \colon \mathcal{X} \to \mathrm{Pro}(\mathbf{Ani})$. Then the *shape* of $\mathcal{X}$ is defined as the image

$$\Pi_\infty(\mathcal{X}) \coloneqq g_\sharp(*_{\mathcal{X}}) \in \mathrm{Pro}(\mathbf{Ani})\,.$$

The *protruncated shape* functor

$$\Pi_{<\infty} \colon \mathbf{RTop} \to \mathrm{Pro}(\mathbf{Ani}_{<\infty})$$

is defined as the composite

$$\mathbf{RTop} \xrightarrow{\Pi_\infty} \mathrm{Pro}(\mathbf{Ani}) \xrightarrow{\mathrm{Pro}(\tau_{<\infty})} \mathrm{Pro}(\mathbf{Ani}_{<\infty})$$

---

[3] This follows from the fact that the homotopy groups of the geometric realization of a Kan complex are computed as its simplicial homotopy groups, and these commute with infinite products.

of the shape with the unique cofiltered limit preserving pro-extension of the functor

$$\tau_{<\infty} : \mathbf{Ani} \to \mathrm{Pro}(\mathbf{Ani}_{<\infty}), \quad A \mapsto \{\tau_{\leq n} A\}_n$$

assigning to an anima its *Postnikov tower*. Similarly, the *profinite shape* is defined by composing further with the profinite completion functor

$$\widehat{\Pi}_\infty : \mathbf{RTop} \xrightarrow{\Pi_{<\infty}} \mathrm{Pro}(\mathbf{Ani}_{<\infty}) \xrightarrow{(-)^{\wedge}_{\pi}} \mathrm{Pro}(\mathbf{Ani}_{\pi})\,.$$

2.24 Notation. For a topological space T, we write $\Pi_\infty(T) \in \mathrm{Pro}(\mathbf{Ani})$ for the shape of the ∞-topos $\mathrm{Sh}(T)$ of sheaves of anima on T. We write $\Pi_{<\infty}(T)$ for the protruncation of $\Pi_\infty(T)$.

We write $\mathbf{LCH} \subset \mathbf{Top}$ for the full subcategory spanned by the locally compact Hausdorff spaces.

2.25 Remark. If T is a topological space that admits a CW structure, then $\Pi_\infty(T)$ coincides with the underlying anima of T.

2.26 Lemma. *The triangle*

$$\begin{tikzcd}
& \mathbf{LCH} \arrow[dl, hook'] \arrow[dr, "\Pi_{<\infty}"] & \\
\mathrm{Cond}(\mathbf{Ani}) \arrow[rr, "(-)^{\wedge}_{\mathrm{disc}}"'] & & \mathrm{Pro}(\mathbf{Ani}_{<\infty})
\end{tikzcd}$$

canonically commutes.

Proof. Let T be a locally compact Hausdorff space. By [32, Corollary 4.9], there is a natural fully faithful left exact left adjoint

$$\mathrm{Sh}^{\mathrm{post}}(T) \hookrightarrow \mathrm{Cond}(\mathbf{Ani})_{/T}$$

from the Postnikov completion of the ∞-topos of sheaves on T to condensed anima sliced over T. Since the protruncated shapes of an ∞-topos and its Postnikov completion coincide, we deduce that this algebraic morphism induces an equivalence on protruncated shapes

$$\Pi_{<\infty}(\mathrm{Cond}(\mathbf{Ani})_{/T}) \xrightarrow{\sim} \Pi_{<\infty}(T)$$

Finally, observe that the protruncated shape of the slice exactly coincides with prodiscrete completion of the condensed set T. ☐

2.27 Remark. Lemma 2.26 was also observed in [4, Theorem 4.12].

2.5 Recollection on proétale sheaves

We now turn to recalling some background about the proétale topology and proétale sheaves. The following definition is from [10]:

2.28 Definition. Let $f : X \to Y$ be a morphism of schemes.

(1) We call $f : X \to Y$ *weakly étale*, if both f and its diagonal Δ_f are flat.

(2) We write $\mathrm{Pro\acute{E}t}_X$ for the *proétale site of* X, i.e., the site of weakly étale X-schemes equipped with the fpqc topology.

(3) We furthermore write $X_{\text{proét}} \coloneqq \mathrm{Sh}(\mathrm{ProÉt}_X)$ for the *proétale ∞-topos of X*.

2.29. We almost exclusively work with the *hypercomplete* proétale ∞-topos $X^{\text{hyp}}_{\text{proét}}$.

2.30 Remark (size issues)**.** Since the category of weakly étale X-schemes is not small, Definition 2.28 introduces some set-theoretic issues. In the end, one can always circumvent these issues and they do not have any serious effect on our results. For the more cautious reader, we suggest one of the following two ways of reading this paper:

(1) Fix once and for all two strongly inaccessible cardinals $\delta < \varepsilon$. All schemes, spectral spaces, etc. are then assumed to be δ-small and all categorical constructions, such as taking sheaves on a site, are taken with respect to the larger universe determined by ε. In particular $X^{\text{hyp}}_{\text{proét}}$ always means hypersheaves of ε-small anima on δ-small weakly étale X-schemes, and similarly for the ∞-category of condensed anima Cond(**Ani**).

(2) If the reader does not want to work with universes, they may proceed as follows. For a scheme X, choose a strong limit cardinal κ such that X is κ-small. Write $\mathrm{ProÉt}_{X,\kappa}$ for the category of κ-small weakly étale X-schemes. We then define

$$X^{\text{hyp}}_{\text{proét},\kappa} \coloneqq \mathrm{Sh}^{\text{hyp}}(\mathrm{ProÉt}_{X,\kappa})\,.$$

The assumption that κ is a strong limit cardinal guarantees that there are enough w-contractibles in $\mathrm{ProÉt}_{X,\kappa}$, see Definition 2.35. We then define

$$X^{\text{hyp}}_{\text{proét}} \coloneqq \mathrm{colim}_\kappa\, X^{\text{hyp}}_{\text{proét},\kappa}$$

and similarly for the category of condensed anima. This is also the approach taken by Clausen and Scholze [70].

However, then some statements about $X^{\text{hyp}}_{\text{proét}}$ and Cond(**Ani**), such as Proposition 2.44, are no longer true on the nose. In such a case, to correct the result, we make an implicit choice of strong limit cutoff cardinal κ, and $X^{\text{hyp}}_{\text{proét}}$ is to be understood as $X^{\text{hyp}}_{\text{proét},\kappa}$. In the end, a choice of such a κ is harmless and does not affect our results, see Remark 3.17.

The same discussion applies to the non-hypercomplete proétale ∞-topos $X_{\text{proét}}$.

We now prove a generalization of [10, Lemma 5.1.2 & Corollary 5.1.6].

2.31 Notation. For a scheme X, we denote the inclusion $\mathrm{Ét}_X \to \mathrm{ProÉt}_X$ of the the étale site into the proétale site by ν.

2.32 Proposition. *Let X be a qcqs scheme. Then the pullback functor $\nu^* : X_{\text{ét}} \to X_{\text{proét}}$ is fully faithful when restricted to truncated objects.*

Proof. First observe that since the left exact pullback functor ν^* preserves n-truncated objects [HTT, Proposition 5.5.6.16], the truncated pullback functors are well-defined. Furthermore, for an n-truncated proétale sheaf F, by [41, Proposition A.1] the sheaf condition can be stated as follows:

(1) The presheaf F sends finite disjoint unions of affine schemes proétale over X to finite products.

(2) For every surjection $f : U \twoheadrightarrow X$ of affine schemes proétale over X with associated Čech nerve $U_\bullet \to X$, the canonical map

$$F(X) \to \lim_{[i]\in\Delta_{\leq n+1}} F(U_i)$$

is an isomorphism.

This is just the n-truncation of the sheaf condition as formulated in [SAG, Proposition A.3.3.1]. (One easily checks that the category $\mathrm{ProÉt}_X^{\mathrm{aff}} \subset \mathrm{ProÉt}_X$ of affine proétale schemes over X, which forms a basis for the proétale topology, satisfies the conditions stated there.)

Since the problem is local on X, we reduce to the case that X is affine. Then, the category $\mathrm{ProÉt}_X^{\mathrm{aff}}$ is exactly given by those $U \in \mathrm{ProÉt}_X$ which can be written as a small cofiltered limit $U = \lim_{i\in I} U_i$ of affine schemes $U_i \in \mathrm{Ét}_X$. For some $n \geq 0$, let F now be an object of $X_{\text{ét},\leq n}$. The presheaf pullback of F to the proétale site of X is given by the formula $U \mapsto \mathrm{colim}_{i\in I^{\mathrm{op}}} F(U_i)$ on all $U \in \mathrm{ProÉt}_X^{\mathrm{aff}}$. We wish to show, that this is already a sheaf. For this, we can just copy the proof of [55, Proposition 7.1.3(2)]. The argument there works not only for equalizers, but for all finite limits as they appear in our n-truncated sheaf condition. As ν^*F restricts to F on affine étale schemes $\mathrm{Ét}_X^{\mathrm{aff}}$, it is clear that we have $\nu_*\nu^*F = F$ for all $F \in X_{\text{ét},\leq n}$, i.e., the pullback $\tau_{\leq n}\,\nu^*$ is fully faithful when restricted to n-truncated objects. See [56, Proposition A.5.33] for more details. ☐

2.33 Notation. Let X be a scheme. We denote by $\Pi^{\text{ét}}_{<\infty}(X)$ the *protruncated étale shape* $\Pi_{<\infty}(X^{\mathrm{hyp}}_{\text{ét}})$ and by $\widehat{\Pi}^{\text{ét}}_{\infty}(X)$ the *profinite étale shape* $\widehat{\Pi}_{\infty}(X^{\mathrm{hyp}}_{\text{ét}})$.

2.34 Corollary. *Let X be a scheme. Then ν induces an equivalence $\Pi_{<\infty}(X^{\mathrm{hyp}}_{\text{proét}}) \to \Pi_{<\infty}(X^{\mathrm{hyp}}_{\text{ét}})$.*

Proof. Immediate from Proposition 2.32 and [8, Example 4.2.8]. ☐

Basis of weakly contractible objects

Recall that an object Y of a site $\mathcal{C}$ is *weakly contractible* if every covering $U \twoheadrightarrow Y$ admits a section. In the proétale site, weakly contractible qcqs objects are given by *w-contractible* schemes.

2.35 Definition. A qcqs scheme X is *w-contractible* if every weakly étale surjection $U \twoheadrightarrow X$ has a section.

For the subsequent characterization of w-contractibles, recall the following fact on connected components of qcqs schemes.

2.36 Lemma [STK, Tag 0900]**.** *Let X be a qcqs scheme. Then $\pi_0(X)$, endowed with the quotient topology induced by $|X|$, is a profinite set.*

2.37 Definition. Let X be a qcqs scheme. We say that X is *w-local* if the subspace $X_{\mathrm{cl}} \subset |X|$ of closed points is closed and every connected component of X has a unique closed point. We stay that X is *w-strictly local* if X is *w-local* and every étale surjection $U \twoheadrightarrow X$ admits a section.

2.38 Remark. As observed in [6, Proposition 3.1], since a w-strictly local scheme is a retract of an affine scheme, every w-strictly local scheme is affine.

2.39 Remark. By [10, Lemma 2.2.9], a qcqs scheme X is w-strictly local if X is w-local and the local rings at all closed points are strictly henselian.

2.40 Example. Let $\bar{k}$ be a separably closed field. Then any qcqs weakly étale $\bar{k}$-scheme X is w-strictly local. Indeed, such a scheme is zero dimensional and thus, by Serre's cohomological characterization of affineness, affine. By [STK, Tag 092Q], it is therefore a cofiltered limit of finite disjoint unions of $\operatorname{Spec}(\bar{k})$ and hence w-strictly local.

2.41 Recollection [STK, Tag 0982]**.** A scheme X is w-contractible if and only if it is w-strictly local and $\pi_0(X) \in \operatorname{Pro}(\mathbf{Set}_{\mathrm{fin}})$ is extremally disconnected. In particular, w-contractible schemes are affine.

2.42 Notation. For a scheme X, we write $\mathrm{ProÉt}_X^{\mathrm{wc}} \subset \mathrm{ProÉt}_X$ for the full subcategory spanned by the w-contractible schemes.

2.43 Recollection [STK, Tag 0990]**.** The subcategory $\mathrm{ProÉt}_X^{\mathrm{wc}} \subset \mathrm{ProÉt}_X$ is a basis for the proétale topology. But beware that $\mathrm{ProÉt}_X^{\mathrm{wc}}$ is not closed under fiber products in $\mathrm{ProÉt}_X$.

2.44 Proposition. *Let X be a scheme. Restriction along the inclusion of sites $\mathrm{ProÉt}_X^{\mathrm{wc}} \subset \mathrm{ProÉt}_X$ defines an equivalence of hypercomplete ∞-topoi*

$$X_{\mathrm{proét}}^{\mathrm{hyp}} = \mathrm{Sh}^{\mathrm{hyp}}(\mathrm{ProÉt}_X) \xrightarrow{\sim} \mathrm{Sh}^{\mathrm{hyp}}(\mathrm{ProÉt}_X^{\mathrm{wc}}) .$$

Moreover, this ∞-topos can be identified with the ∞-topos of finite product-preserving presheaves

$$\mathrm{Fun}^{\times}((\mathrm{ProÉt}_X^{\mathrm{wc}})^{\mathrm{op}}, \mathbf{Ani}) .$$

Proof. This follows from Recollection 2.43 and [3, Corollary A.7] combined with the defining property of w-contractible schemes. Details are given in [56, Proposition 2.2.12]. ☐

Part I

Foundational results

3 The condensed homotopy type

In this section, we introduce the condensed homotopy type of a scheme X. As explained in the introduction, we give three definitions, and prove that they are equivalent. The first, given in §3.1, is the relative shape of the hypercomplete proétale ∞-topos $X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}}$ over the ∞-topos $\mathrm{Cond}(\mathbf{Ani})$ of condensed anima. The second, given in § 3.2, is as the unique hypercomplete proétale cosheaf whose value on a w-contractible affine U is the profinite set $\pi_0(U)$ of connected components of U. The last, given in § 3.3, is as the condensed classifying anima of the Galois category $\mathrm{Gal}(X)$ introduced by Barwick–Glasman–Haine [8]. In § 3.4, we conclude the section with a sample computation: given a henselian local ring R with residue field κ, we show inclusion of the closed point induces an equivalence

$$\mathrm{BGal}_\kappa \simeq \Pi^{\mathrm{cond}}_\infty(\mathrm{Spec}(\kappa)) \xrightarrow{\sim} \Pi^{\mathrm{cond}}_\infty(\mathrm{Spec}(R)) .$$

3.1 Definition via the relative shape

For an ∞-topos $\mathcal{X}$, the idea of shape theory relies on the existence of a canonical colimit preserving functor $\Gamma_\sharp \colon \mathcal{X} \to \mathrm{Pro}(\mathbf{Ani})$. We define the condensed homotopy type of a qcqs scheme in the tradition of shape theory but relative to the base $\mathrm{Cond}(\mathbf{Ani})$. To do this, we use the identification

$$X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}} \simeq \mathrm{Fun}^\times\left((\mathrm{Pro\acute{E}t}^{\mathrm{wc}}_X)^{\mathrm{op}}, \mathbf{Ani}\right)$$

of the hypercomplete proétale ∞-topos as the ∞-topos of finite-product preserving presheaves on the site of w-contractible weakly étale X-schemes (Proposition 2.44).

3.1 Definition. Let X be a scheme. Write

$$\pi_\sharp \colon \mathrm{PSh}(\mathrm{Pro\acute{E}t}^{\mathrm{wc}}_X) \to \mathrm{Cond}(\mathbf{Ani})$$

for the colimit-preserving extension of

$$\pi_0 \colon \mathrm{Pro\acute{E}t}^{\mathrm{wc}}_X \to \mathbf{Extr} \hookrightarrow \mathrm{Cond}(\mathbf{Ani})$$

along the Yoneda embedding.

3.2 Observation. The functor $\pi_\sharp$ admits a right adjoint

$$\pi^* \colon \mathrm{Cond}(\mathbf{Ani}) \to \mathrm{PSh}(\mathrm{Pro\acute{E}t}^{\mathrm{wc}}_X)$$

given by the assignment

$$A \mapsto [W \mapsto A(\pi_0(W))] .$$

Note that since the functor $\pi_0 \colon \mathrm{Pro\acute{E}t}^{\mathrm{wc}}_X \to \mathrm{Cond}(\mathbf{Ani})$ preserves finite disjoint unions, the right adjoint to $\pi_\sharp$ factors through

$$\mathrm{Fun}^\times\left((\mathrm{Pro\acute{E}t}^{\mathrm{wc}}_X)^{\mathrm{op}}, \mathbf{Ani}\right) \subset \mathrm{PSh}(\mathrm{Pro\acute{E}t}^{\mathrm{wc}}_X) .$$

3.3 Notation. Given a scheme X, we also write $\pi_\sharp$ for the composite

$$X_{\text{proét}}^{\text{hyp}} \xrightarrow{\sim} \mathrm{Fun}^{\times}\left((\mathrm{Pro\acute{E}t}_X^{\mathrm{wc}})^{\mathrm{op}}, \mathbf{Ani}\right) \xrightarrow{\pi_\sharp} \mathrm{Cond}(\mathbf{Ani}) ,$$

where the left-hand functor is the equivalence of ∞-topoi from Proposition 2.44.

Next, we need a generalization of [10, Lemma 4.2.13].

3.4 Proposition. *Let X be a scheme. Then:*

(1) *The functor* $\pi_\sharp : X_{\text{proét}}^{\text{hyp}} \to \mathrm{Cond}(\mathbf{Ani})$ *is left adjoint to* $\pi^* : \mathrm{Cond}(\mathbf{Ani}) \to X_{\text{proét}}^{\text{hyp}}$.

(2) *For each condensed anima A and w-contractible affine $W \in \mathrm{Pro\acute{E}t}_X$, there is a natural equivalence*

$$\pi^*(A)(W) \simeq A(\pi_0(W)) .$$

Proof. As explained in Observation 3.2, the functor

$$\pi^* : \mathrm{Cond}(\mathbf{Ani}) \to \mathrm{PSh}(\mathrm{Pro\acute{E}t}_X^{\mathrm{wc}})$$

factors through $X_{\text{proét}}^{\text{hyp}}$. Hence π^* remains right adjoint to the restriction of $\pi_\sharp$. In particular, we have $\pi^*(A)(U) \simeq A(\pi_0(U))$. ☐

3.5 Remark. The right adjoint π^* is part of a geometric morphism of ∞-topoi

$$\begin{tikzcd} \mathrm{Cond}(\mathbf{Ani}) \arrow[r, shift left, "\pi^*"] & X_{\text{proét}}^{\text{hyp}} \arrow[l, shift left, "\pi_*"] \end{tikzcd} , \tag{3.6}$$

which is induced by the morphism of sites

$$\begin{aligned} \pi : \ \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}}) &\longrightarrow \mathrm{Pro\acute{E}t}_X , \\ S = \lim_{i \in I} S_i &\longmapsto S \otimes X := \lim_{i\in I} \coprod_{s \in S_i} X . \end{aligned}$$

For details, see [56, Theorem 2.2.13].

3.7 Definition. Let X be a scheme.

(1) The *condensed homotopy type* of X is the condensed anima

$$\Pi_\infty^{\mathrm{cond}}(X) := \pi_\sharp(1) \in \mathrm{Cond}(\mathbf{Ani}) .$$

(2) The *condensed set of connected components* of X is the condensed set

$$\pi_0^{\mathrm{cond}}(X) := \pi_0(\Pi_\infty^{\mathrm{cond}}(X)) \in \mathrm{Cond}(\mathbf{Set}) .$$

3.8. The first part of Definition 3.7 says that the condensed homotopy type is the relative shape of the ∞-topos $X_{\text{proét}}^{\text{hyp}}$ over the ∞-topos $\mathrm{Cond}(\mathbf{Ani})$, see [13, §4.1] for background on relative shapes. Since sending a scheme X to $\pi_* : X_{\text{proét}}^{\text{hyp}} \to \mathrm{Cond}(\mathbf{Ani})$ defines a functor

$$\mathbf{Sch} \to (\mathbf{RTop}_\infty)_{/\,\mathrm{Cond}(\mathbf{Ani})} .$$

Composition with the relative shape over $\mathrm{Cond}(\mathbf{Ani})$, therefore defines a functor

$$\Pi_\infty^{\mathrm{cond}} : \mathbf{Sch} \to \mathrm{Cond}(\mathbf{Ani}) , \ X \mapsto \Pi_\infty^{\mathrm{cond}}(X) . \tag{3.9}$$

3.10 Warning. The first point of [10, Lemma 4.2.13] is not true in the stated generality. It says that (for condensed sets A) the formula $\pi^*(A)(U) \simeq A(\pi_0(U))$ in Proposition 3.4 holds for all qcqs schemes U of the proétale site of X. It seems, however, that the proof in *loc. cit.* only works for w-contractible schemes. Indeed, if this stronger claim was true, it would follow that for all qcqs schemes X one has

$$\begin{aligned} \mathrm{Map}_{\mathrm{Cond}(\mathbf{Set})}(\pi_0(X), A) \simeq A(\pi_0(X)) &\simeq \pi^*(A)(X) \\ &\simeq \mathrm{Map}_{X^{\mathrm{hyp}}_{\text{proét}}}(X, \pi^*(A)) \\ &\simeq \mathrm{Map}_{\mathrm{Cond}(\mathbf{Ani})}(\Pi^{\mathrm{cond}}_{\infty}(X), A) \\ &\simeq \mathrm{Map}_{\mathrm{Cond}(\mathbf{Set})}(\pi^{\mathrm{cond}}_{0}(X), A)\,. \end{aligned}$$

This would then imply that the condensed set of connected components matches the usual one, i.e., $\pi^{\mathrm{cond}}_0(X) = \pi_0(X)$ in Cond(**Set**). As we show in Example 4.24, this is not generally the case. However, this is true if X has finitely many irreducible components, see Corollary 4.19.

The definition tells us the value of the condensed homotopy type on w-contractible schemes:

3.11 Example. Let X be a w-contractible scheme. Then, by definition,

$$\Pi^{\mathrm{cond}}_{\infty}(X) = \pi_\sharp(1) = \pi_0(X)\,.$$

In particular, if X is the spectrum of a separably closed field, then $\Pi^{\mathrm{cond}}_{\infty}(X) = *$.

3.12. One consequence of Example 3.11 is that every geometric point $\bar{x} \to X$ defines a point

$$* = \Pi^{\mathrm{cond}}_{\infty}(\bar{x}) \to \Pi^{\mathrm{cond}}_{\infty}(X)$$

of the condensed homotopy type. Thus we can take homotopy groups at geometric points:

3.13 Definition. Let X be a scheme, let $\bar{x} \to X$ be a geometric point, and let $n \geq 1$. The *n-th condensed homotopy group* of X at $\bar{x}$ is the condensed group (abelian if $n \geq 2$)

$$\pi^{\mathrm{cond}}_{n}(X, \bar{x}) \coloneqq \pi_n(\Pi^{\mathrm{cond}}_{\infty}(X), \bar{x})\,.$$

From the definition, it is easy to see that the condensed homotopy type refines the protruncated and profinite étale homotopy types. For this result, recall our notation on shapes and étale homotopy types from §2.4 and Notation 2.33.

3.14 Lemma. *Let X be a scheme. Then there are natural equivalences*

$$\Pi^{\mathrm{cond}}_{\infty}(X)^{\wedge}_{\mathrm{disc}} \simeq \Pi^{\text{ét}}_{<\infty}(X) \qquad \textit{and} \qquad \Pi^{\mathrm{cond}}_{\infty}(X)^{\wedge}_{\pi} \simeq \widehat{\Pi}^{\text{ét}}_{\infty}(X)\,.$$

Proof. By Corollary 2.34, the protruncated shapes of the (hypercomplete) étale and proétale ∞-topoi agree. This remains true after profinite completion. Thus the claims follow from the claim that the triangle of left adjoints

$$\begin{tikzcd} & X^{\mathrm{hyp}}_{\text{proét}} \arrow[dl, "\pi_\sharp"'] \arrow[dr, "\Pi^{\text{ét}}_{<\infty}"] & \\ \mathrm{Cond}(\mathbf{Ani}) \arrow[rr, "(-)^{\wedge}_{\mathrm{disc}}"'] & & \mathrm{Pro}(\mathbf{Ani}_{<\infty}) \end{tikzcd}$$

commutes. To see this, note that the corresponding diagram of right adjoints commutes by the uniqueness property of the pro-extension $\mathrm{Pro}(\mathbf{Ani}) \to X^{\mathrm{hyp}}_{\text{proét}}$ of the constant sheaf functor. ☐

3.2 Characterization as a hypercomplete proétale cosheaf

The goal of this subsection is to prove the following characterization of the condensed homotopy type and derive some consequences for the étale homotopy type.

3.15 Notation. We write $\mathbf{Aff}^{\mathrm{wc}} \subset \mathbf{Sch}$ for the full subcategory spanned by the w-contractible schemes. (Recall from Recollection 2.41 that w-contractible schemes are affine.)

3.16 Proposition. *The condensed homotopy type*

$$\Pi_\infty^{\mathrm{cond}} \colon \mathbf{Sch} \to \mathrm{Cond}(\mathbf{Ani})$$

is the unique hypercomplete proétale cosheaf whose restriction to w-contractible schemes is given by the functor

$$\pi_0 \colon \mathbf{Aff}^{\mathrm{wc}} \to \mathbf{Extr} \subset \mathrm{Cond}(\mathbf{Ani})\,.$$

Proof. First notice that since $\pi_\sharp$ preserves colimits, by definition $\Pi_\infty^{\mathrm{cond}}$ carries proétale hypercoverings to colimit diagrams. Moreover, by construction $\Pi_\infty^{\mathrm{cond}}$ agrees with π_0 when restricted to w-contractible schemes (see Example 3.11). Thus it suffices to show that every scheme admits a proétale hypercover by w-contractible schemes. Since every scheme admits a Zariski cover by qcqs schemes, we can reduce to the qcqs case. In this case, the claim is the content of [STK, Tag 09A1]. ☐

3.17 Remark (on set theory)**.** Let X be a scheme and κ a strong limit cardinal such that X is κ-small. Then there exists a hypercover by w-contractibles $W_\bullet \to X$ such that W_n is κ-small for all n. Hence the formula

$$\Pi_\infty^{\mathrm{cond}}(X) \simeq \mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}}\, \pi_0(W_\bullet)$$

shows that for $\kappa < \kappa'$ an implicit choice of cutoff cardinal in Definition 3.7 does not affect the outcome. More precisely, under the embedding $\mathrm{Cond}(\mathbf{Ani})_\kappa \hookrightarrow \mathrm{Cond}(\mathbf{Ani})_{\kappa'}$ one gets carried to the other. Equivalently, if one takes the approach to dealing with set theory explained in Remark 2.30 (2), then for all choices of suitable cutoff cardinals the images of the condensed homotopy type in the colimit $\mathrm{Cond}(\mathbf{Ani}) = \mathrm{colim}_\kappa\, \mathrm{Cond}(\mathbf{Ani})_\kappa$ agree. Therefore we can continue to leave choices of cutoff cardinals implicit without getting into trouble.

If one would try to set up the theory in the setting of *light* condensed anima, one would get a different result in general. See also Remark 3.41.

3.18 Corollary.

(1) *The functor*

$$\Pi_{<\infty}^{\mathrm{\acute{e}t}} \colon \mathbf{Sch} \to \mathrm{Pro}(\mathbf{Ani}_{<\infty})$$

is the unique hypercomplete proétale cosheaf valued in $\mathrm{Pro}(\mathbf{Ani}_{<\infty})$ *whose restriction to w-contractible affines coincides with*

$$\pi_0 \colon \mathbf{Aff}^{\mathrm{wc}} \to \mathbf{Extr} \hookrightarrow \mathrm{Pro}(\mathbf{Ani}_{<\infty})\,.$$

(2) *The functor*

$$\widehat{\Pi}_\infty^{\mathrm{\acute{e}t}} \colon \mathbf{Sch} \to \mathrm{Pro}(\mathbf{Ani}_\pi)$$

is the unique hypercomplete proétale cosheaf valued in $\mathrm{Pro}(\mathbf{Ani}_\pi)$ *whose restriction to w-contractible affines coincides with*

$$\pi_0 \colon \mathbf{Aff}^{\mathrm{wc}} \to \mathbf{Extr} \hookrightarrow \mathrm{Pro}(\mathbf{Ani}_\pi)\,.$$

Proof. Since both $(-)^{\wedge}_{\mathrm{disc}}$ and $(-)^{\wedge}_{\pi}$ are left adjoints, the composites

$$\begin{tikzcd} \mathbf{Sch} \arrow[r, "\Pi_\infty^{\mathrm{cond}}"] & \mathrm{Cond}(\mathbf{Ani}) \arrow[r, "(-)^{\wedge}_{\mathrm{disc}}"] & \mathrm{Pro}(\mathbf{Ani}_{<\infty}) \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbf{Sch} \arrow[r, "\Pi_\infty^{\mathrm{cond}}"] & \mathrm{Cond}(\mathbf{Ani}) \arrow[r, "(-)^{\wedge}_{\pi}"] & \mathrm{Pro}(\mathbf{Ani}_{\pi}) \end{tikzcd}$$

are still hypercomplete proétale cosheaves. Moreover, on w-contractible affines they both are given by $U \mapsto \pi_0(U) \in \mathbf{Extr}$. In Lemma 3.14, we have seen that these functors recover the protruncated and profinite étale homotopy types, respectively. ☐

3.19 Remark. It follows immediately from Proposition 3.16 that the 'condensed shape' defined in [40, Appendix A] agrees with our notions.

In [40], Hemo–Richarz–Scholbach prove that $\Pi_\infty^{\mathrm{cond}}(X)$ classifies local systems on X with coefficients in any condensed ring. We recall the precise statement here; for this, we need the following definition from [40]. In order to state it, recall that we write π^* for the natural pullback functor $\mathrm{Cond}(\mathbf{Ani}) \to X^{\mathrm{hyp}}_{\mathrm{proét}}$ of Observation 3.2.

3.20 Definition. Let Λ be a condensed ring.

(1) We define the condensed ∞-category $\mathbf{Perf}_\Lambda$ of *perfect complexes* over Λ, to be the condensed ∞-category defined by

$$\begin{aligned} \mathbf{Extr}^{\mathrm{op}} &\to \mathbf{Cat}_\infty \\ S &\mapsto \mathrm{Perf}_{\Lambda(S)} \ . \end{aligned}$$

Here, $\mathrm{Perf}_{\Lambda(S)}$ is the usual ∞-category of perfect complexes over the ordinary ring $\Lambda(S)$.

(2) Let X be a qcqs scheme. Write $\mathrm{D}(X_{\mathrm{proét}};\Lambda)$ for the derived ∞-category of $\pi^*\Lambda$-modules on X. We define the ∞-category of *lisse* Λ-modules $\mathrm{D}_{\mathrm{lis}}(X_{\mathrm{proét}};\Lambda)$ to be the full subcategory of $\mathrm{D}(X_{\mathrm{proét}};\Lambda)$ spanned by the dualizable objects.

3.21 Proposition [40, Proposition A.1]**.** *There is a natural equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{cts}}(\Pi_\infty^{\mathrm{cond}}(X), \mathbf{Perf}_\Lambda) \simeq \mathrm{D}_{\mathrm{lis}}(X_{\mathrm{proét}};\Lambda) \, .$$

3.22 Remark. Proposition 3.21 is one of the main motivations to study the condensed homotopy type. Indeed, the analogous statement for the ususal étale homotopy type $\Pi_\infty^{\mathrm{ét}}(X)$ is not even true in for $\Lambda = \mathbf{Q}_\ell$. See [10, Example 7.4.9] for a concrete counterexample.

3.3 Definition via exodromy

In this subsection, we explain why the *pyknotic étale homotopy type* defined in [8, Remark 13.8.10] agrees with $\Pi_\infty^{\mathrm{cond}}(X)$. For this, we recall the following definition from [8] in the general setting of coherent ∞-topoi, but we are most interested in the case of the étale ∞-topos of a scheme. In order to understand the general definition, the reader may wish to review the theory of coherent ∞-topoi from [SAG, Appendix A] or [8, Chapter 3].

3.23 Definition. Let $\mathcal{X}$ be a coherent ∞-topos. The *Galois ∞-category* of $\mathcal{X}$ is the condensed ∞-category $\mathrm{Gal}(\mathcal{X})$ defined by the functor

$$\begin{aligned} \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})^{\mathrm{op}} &\to \mathbf{Cat}_{\infty} \\ S &\mapsto \mathrm{Fun}^{*,\mathrm{coh}}(\mathcal{X}, \mathrm{Sh}(S))\,. \end{aligned}$$

Here, $\mathrm{Fun}^{*,\mathrm{coh}}(\mathcal{X}, \mathrm{Sh}(S))$ is the ∞-category of *coherent* algebraic morphisms $s^* \colon \mathcal{X} \to \mathrm{Sh}(S)$ of ∞-topoi, i.e., those left exact left adjoints that send truncated coherent objects of $\mathcal{X}$ to locally constant constructible sheaves of anima on the topological space S.

The assignment $\mathcal{X} \mapsto \mathrm{Gal}(\mathcal{X})$ defines a functor from the ∞-category of coherent ∞-topoi and coherent geometric morphisms to $\mathrm{Cond}(\mathbf{Cat}_{\infty})$.

Now we explain what this definition means more concretely in the two examples we are interested in in this paper.

3.24 Recollection. Let X be a qcqs scheme. Then the ∞-topos $X_{\text{ét}}$ is coherent and by [8, Lemma 9.5.3 & Proposition 9.5.4], the truncated coherent objects of $X_{\text{ét}}$ are the constructible étale sheaves of anima on X.

3.25 Notation. Let X be a qcqs scheme. We write $\mathrm{Gal}(X) \coloneqq \mathrm{Gal}(X_{\text{ét}})$.

3.26 Recollection. Let X be a qcqs scheme. Since the ∞-topos $X_{\text{ét}}$ is 1-localic, for a profinite set S, the value $\mathrm{Gal}(X)(S)$ is equivalent to the 1-category of algebraic morphisms of 1-topoi

$$s^* \colon X_{\text{ét},\leq 0} \to \mathrm{Sh}(S)_{\leq 0}$$

that send constructible étale sheaves of sets to locally constant constructible sheaves of sets on S. In particular, the global sections $\mathrm{Gal}(X)(*)$ recovers the category of points of the étale topos of X.

3.27 Recollection. Let T be a spectral space (e.g., the underlying space of a qcqs scheme). Then the ∞-topos $\mathrm{Sh}(T)$ is coherent and by [8, Lemma 9.5.3 & Proposition 9.5.4], the truncated coherent objects of $\mathrm{Sh}(T)$ are the constructible sheaves of anima on T.

3.28 Notation. For a spectral space T, we write $\mathrm{Gal}(T_{\mathrm{zar}}) \coloneqq \mathrm{Gal}(\mathrm{Sh}(T))$.

3.29 Recollection. Let T be a spectral space. Since spectral spaces are sober, by [8, Example 3.7.1] and [HTT, Remark 6.4.5.3], for a profinite set S, the value $\mathrm{Gal}(T_{\mathrm{zar}})(S)$ is equivalent to the *poset* of quasicompact maps $f \colon S \to T$ ordered by *pointwise specialization: $f \leq g$* if and only if for all $s \in S$, we have $f(s) \in \overline{\{g(s)\}}$. In particular, $\mathrm{Gal}(T_{\mathrm{zar}})(*)$ recovers the specialization poset of T.

3.30 Remark. Note that the condensed set underlying the condensed poset $\mathrm{Gal}(T_{\mathrm{zar}})$ is indeed a condensed set, i.e., is κ-accessible for some κ. In contrast, the condensed set represented by the topological space T is typically not κ-accessible, see [70, Warning 2.14]. The difference between the two is that $\mathrm{Gal}(T_{\mathrm{zar}})(S)$ is given by the set of *quasicompact* maps $S \to T$, as opposed to all continuous maps.

3.31 Recollection. For a qcqs scheme X, the condensed ∞-categories $\mathrm{Gal}(X)$ and $\mathrm{Gal}(X_{\mathrm{zar}})$ are in the image of the fully faithful functor

$$\iota \colon \mathrm{Cat}(\mathrm{Pro}(\mathbf{Ani}_{\pi})) \to \mathrm{Cond}(\mathbf{Cat}_{\infty})$$

of Observation 2.21. In fact, if we denote by $\mathbf{Lay}_\pi$ the full subcategory of $\mathbf{Cat}_\infty$ spanned by π-finite layered categories in the sense of [8, Definition 2.3.7], then $\mathrm{Gal}(X)$ and $\mathrm{Gal}(X_{\mathrm{zar}})$ are even in the image of the fully faithful functor $\mathrm{Pro}(\mathbf{Lay}_\pi) \to \mathrm{Cond}(\mathbf{Cat}_\infty)$. See [8, §13.5] for more details.

Now we fix some notation regarding condensed ∞-categories and classifying anima.

3.32 Definition. We define condensed ∞-categories $\mathbf{Cond}(\mathbf{Ani})$ and $\mathbf{Cond}(\mathbf{Set})$ by the assignments

$$S \mapsto \mathrm{Cond}(\mathbf{Ani})_{/S} \qquad \text{and} \qquad S \mapsto \mathrm{Cond}(\mathbf{Set})_{/S}\ ,$$

respectively.

3.33 Notation. We denote the left adjoint to the inclusion $\mathbf{Ani} \hookrightarrow \mathbf{Cat}_\infty$ by $\mathrm{B} \colon \mathbf{Cat}_\infty \to \mathbf{Ani}$. Given an ∞-category $\mathcal{C}$, we call $\mathrm{B}\mathcal{C}$ the *classifying anima* of $\mathcal{C}$.

3.34. The functor B preserves finite products. Hence post-composition with B induces a functor

$$\mathrm{B}^{\mathrm{cond}} \colon \mathrm{Cond}(\mathbf{Cat}_\infty) \to \mathrm{Cond}(\mathbf{Ani})$$

that is left adjoint to the inclusion $\mathrm{Cond}(\mathbf{Ani}) \hookrightarrow \mathrm{Cond}(\mathbf{Cat}_\infty)$.

3.35 Definition. Given a condensed ∞-category $\mathcal{C}$, we call $\mathrm{B}^{\mathrm{cond}}(\mathcal{C}) \in \mathrm{Cond}(\mathbf{Ani})$ the *condensed classifying anima* of $\mathcal{C}$.

To see the desired comparison, the idea is that, by [80, Corollary 1.2], we have a natural equivalence

$$\mathrm{Fun}^{\mathrm{cts}}(\mathrm{Gal}(X), \mathbf{Cond}(\mathbf{Ani})) \simeq X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}}\ .$$

In other words, in the condensed world, $X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}}$ is a presheaf ∞-category on $\mathrm{Gal}(X)^{\mathrm{op}}$. But the shape of a presheaf ∞-topos is given by taking the classifying anima of the ∞-category that it is presheaves on; the same holds in the condensed world.

3.36 Proposition. *Let X be a qcqs scheme. Then there is a natural equivalence of condensed anima*

$$\Pi^{\mathrm{cond}}_\infty(X) \simeq \mathrm{B}^{\mathrm{cond}}\mathrm{Gal}(X)\ .$$

Proof. This is an immediate consequence of [80, Theorem 1.2] and [59, Proposition 4.4.1]. For the reader not so familiar with the theory developed in [59], we spell out a more hands-on proof. Recall that for ∞-categories $\mathcal{C}$ and $\mathcal{D}$, the functor

$$\mathrm{Fun}(\mathrm{B}\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

induced by precomposition along $\mathcal{C} \to \mathrm{B}\mathcal{C}$ is fully faithful (since $\mathrm{B}\mathcal{C} \simeq \mathcal{C}[\mathcal{C}^{-1}]$ is the localization of $\mathcal{C}$ obtained by inverting all maps, this follows from the universal property of localization). Since limits of fully faithful functors are fully faithful [37, Proposition 2.1; 56, Proposition A.1.3], it follows that precomposition with $b \colon \mathrm{Gal}(X) \to \mathrm{B}^{\mathrm{cond}}\mathrm{Gal}(X)$ defines a fully faithful functor

$$\mathrm{Fun}^{\mathrm{cts}}(\mathrm{B}^{\mathrm{cond}}\mathrm{Gal}(X), \mathbf{Cond}(\mathbf{Ani})) \xrightarrow{b^*} \mathrm{Fun}^{\mathrm{cts}}(\mathrm{Gal}(X), \mathbf{Cond}(\mathbf{Ani}))\ .$$

Furthermore, by [80, Lemma 4.3] this functor admits a left adjoint $b_\sharp$.

By [80, Corollary 1.2] we have a natural equivalence $X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}} \simeq \mathrm{Fun}^{\mathrm{cts}}(\mathrm{Gal}(X), \mathbf{Cond}(\mathbf{Ani}))$. Under this equivalence the functor

$$\pi^* \colon \mathrm{Cond}(\mathbf{Ani}) \to X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}}$$

agrees with the functor given by precomposing with the unique morphism $\mathrm{Gal}(X) \to *$. We write $a \colon \mathrm{B}^{\mathrm{cond}}\mathrm{Gal}(X) \to *$ for the unique morphism, and obtain a commutative triangle

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{cts}}(\mathrm{B}^{\mathrm{cond}}\mathrm{Gal}(X), \mathbf{Cond}(\mathbf{Ani})) \arrow[r, "b^*"] & X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}} \\
\mathrm{Cond}(\mathbf{Ani}) \arrow[u, "a^*"] \arrow[ur, "\pi^*"'] & .
\end{tikzcd}$$

But now since b^* is fully faithful and $b^*(1) = 1$, it follows that $b_\sharp(1) = 1$, Thus,

$$\pi_\sharp(1) = a_\sharp b_\sharp(1) = a_\sharp(1)\,.$$

Finally, by [80, Corollary 3.20] we have

$$\mathrm{Fun}^{\mathrm{cts}}(\mathrm{B}^{\mathrm{cond}}\mathrm{Gal}(X), \mathbf{Cond}(\mathbf{Ani})) \simeq \mathrm{Cond}(\mathbf{Ani})_{/\mathrm{B}^{\mathrm{cond}}\mathrm{Gal}(X)}$$

and the functor $a_\sharp$ identifies with the forgetful functor. In particular $a_\sharp(1) \simeq \mathrm{B}^{\mathrm{cond}}\mathrm{Gal}(X)$. ☐

3.37 Corollary. *Let X be a qcqs scheme. If* $\dim(X) = 0$, *then* $\Pi_\infty^{\mathrm{cond}}(X) = \mathrm{Gal}(X)$ *and this condensed anima is a* 1*-truncated profinite anima.*

Proof. This is immediate from [36, Observation 1.25] and Recollection 3.31. ☐

3.38 Example ($\Pi_\infty^{\mathrm{cond}}$ of a field)**.** Let k be a field and choose a separable closure $\bar{k}$ of k. Write Gal_k for the absolute Galois group of k with respect to $\bar{k}$. Then the choice of separable closure induces an equivalence

$$\Pi_\infty^{\mathrm{cond}}(\mathrm{Spec}(k)) = \mathrm{Gal}(\mathrm{Spec}(k)) \simeq \mathrm{BGal}_k\ .$$

The left-hand identification follows from Corollary 3.37, and the right-hand identification follows from [8, Examples 11.2.1 and 12.2.1].

We do not use the next corollary in the remainder of this article, but we include it for completeness:

3.39 Corollary. *Let X be a qcqs scheme. If* $\dim(X) = 0$, *then* $\Pi_\infty^{\mathrm{cond}}(X) = *$ *if and only if the reduced scheme X_{red} is* $\mathrm{Spec}(k)$ *for k a separably closed field.*

Proof. As the étale ∞-topos is invariant under universal homeomorphisms, the same holds for Gal and therefore $\Pi_\infty^{\mathrm{cond}}$. As $X \to X_{\mathrm{red}}$ is a universal homeomorphism, the if direction follows by the Example 3.38. For the reverse direction, note that $\mathrm{Pt}(X_{\mathrm{\acute{e}t}})$ of a 0-dimensional affine scheme is contractible only if $X = \mathrm{Spec}(R)$ for R a local ring with separably closed residue field k. For such a scheme, it is $X_{\mathrm{red}} = \mathrm{Spec}(k)$. ☐

3.4 Computation: $\Pi_\infty^{\mathrm{cond}}$ of henselian local rings

We conclude this section by explaining how to use the definitions to show that the condensed homotopy type of a w-strictly local scheme X (in the sense of Definition 2.37) agrees with the profinite set $\pi_0(X)$ of connected components of X. This allows for a direct computation of the condensed homotopy type of a henselian local ring.

3.40 Proposition. *Let X be a w-strictly local scheme. Then $\Pi_\infty^{\mathrm{cond}}(X) \simeq \pi_0(X)$.*

3.41 Remark. Let X be a qcqs scheme that locally can be written as the spectrum of a countable colimit of finite type $\mathbf{Z}$-algebras. Then one can show that there is a hypercover $W_\bullet \to X$ consisting of w-strictly local X-schemes with the property that $\pi_0(X)$ is a light condensed set. Hence it follows from Proposition 3.40 that in this case $\Pi_\infty^{\mathrm{cond}}(X)$ is a light condensed anima in the sense that it is in the image of the fully faithful functor

$$\mathrm{Sh}(\mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})_{\aleph_1}) \hookrightarrow \mathrm{Cond}(\mathbf{Ani})\,.$$

For a general scheme X, the condensed homotopy type $\Pi_\infty^{\mathrm{cond}}(X)$ need not be light.

Recall that the proétale site is “tensored” over profinite sets (cf. [10, Example 4.1.9]).

3.42 Lemma. *Let X be an affine scheme and $f_0 : S \to \pi_0(X)$ a map from a profinite set. Let $X' =$ "$X \otimes_{\pi_0(X)} S$" be the affine scheme constructed (functorially) in [10, Lemma 2.2.8] with a proétale map $f : X' \to X$ satisfying $\pi_0(f) = f_0$. If X is w-strictly local, then so is X'.*

Proof. We can split the construction of X' into two steps: first consider $X'' = X \otimes S$ coming from "tensoring" by S. It satisfies $\pi_0(X'') = \pi_0(X) \times S$. Then realize X' as a closed subscheme of X'' that is moreover an intersection of clopen subschemes, by looking at $S \subset \pi_0(X) \times S = \pi_0(X'')$ and writing S as an intersection of clopen subsets in this larger set.

Let us first check it for X''. By definition and [10, Lemma 2.2.9], an affine scheme is w-strictly local if it is w-local and all of its connected components are spectra of strictly henselian rings. Here, we are using the following observation: the connected components of a w-local affine scheme are spectra of local rings. Indeed, they are affine (being closed subschemes of an affine scheme) and have a single closed point (by definition of w-locality). Thus, Zariski localizations at closed points of a w-local affine scheme match the corresponding connected components.

One checks that both of these conditions are satisfied for $X'' = X \otimes S$ by checking the following facts: $\pi_0(X \otimes S) = \pi_0(X) \times S$, every connected component of $X \otimes S$ is isomorphic (as a scheme) to some connected component of X, and $(X \otimes S)_{\mathrm{cl}} \simeq X_{\mathrm{cl}} \otimes S$. Each of those is reasonably easy to check, as $X \otimes S$ is defined as an inverse limit of the form $\lim_i X^{S_i} = \lim_i (X \sqcup \ldots \sqcup X)$ where the transition maps restricted to each copy of X appearing there are just identities onto another copy of X. Here $S = \lim_i S_i$ for finite sets S_i.

The second step of passing from X'' to X' by intersecting an inverse system of clopen subschemes follows in a similar way. ☐

Proof of Proposition 3.40. By Proposition 3.16, this statement holds when X is w-contractible. In general, pick a hypercover of the profinite set $\pi_0(X)$ by extremally disconnected profinite sets. By [10, Lemma 2.2.8], Recollection 2.41, and Lemma 3.42, we obtain a proétale hypercover $X_\bullet \to X$ by w-contractible affine schemes[4] that recovers the original hypercover of $\pi_0(X)$ after applying π_0. We compute

$$\begin{aligned}\Pi_\infty^{\mathrm{cond}}(X) &\simeq \operatorname*{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \Pi_\infty^{\mathrm{cond}}(X_n) \\ &\simeq \operatorname*{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \pi_0(X_n) \simeq \pi_0(X)\,,\end{aligned}$$

as desired. ☐

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[4] Here we have used that the functor in *loc. cit.* commutes with limits and respects covers.

We now move on to the promised applications.

3.43 Corollary. *Let S be a profinite set and X a w-strictly local scheme. Then*

$$\Pi_\infty^{\mathrm{cond}}(X \otimes S) \simeq \pi_0(X) \times S\,.$$

Proof. This follows from Proposition 3.40 and Lemma 3.42 with $f_0 = \mathrm{pr}_1 : \pi_0(X) \times S \to \pi_0(X)$ together with the equality $\pi_0(X \otimes S) = \pi_0(X) \times S$. ☐

3.44 Corollary. *Let R be a henselian local ring with residue field κ. Then the inclusion of the closed point* $\mathrm{Spec}(\kappa) \hookrightarrow \mathrm{Spec}(R)$ *induces an equivalence*

$$\Pi_\infty^{\mathrm{cond}}(\mathrm{Spec}(\kappa)) \xrightarrow{\sim} \Pi_\infty^{\mathrm{cond}}(\mathrm{Spec}(R))$$

and both are equivalent to BGal_κ.

Proof. Write $X = \mathrm{Spec}(R)$ and $x = \mathrm{Spec}(\kappa)$. Fix a separable closure $\overline{\kappa}$ of κ and let R^{sh} be the corresponding strict henselization. Writing $\overline{\kappa}$ as an increasing union of finite separable extensions (and using that $\mathrm{F\acute{E}t}_x \simeq \mathrm{F\acute{E}t}_X$) provides a presentation of $X' = \mathrm{Spec}(R^{\mathrm{sh}})$ as a pro-(finite étale) cover of X, see [STK, Tag 0BSL]. Let $X_\bullet$ be the Čech nerve of this cover $X' \to X$. As the equivalence $\mathrm{F\acute{E}t}_x \simeq \mathrm{F\acute{E}t}_X$ extends to the categories of pro-objects, we compute that $X_\bullet$ writes as

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X' \otimes \mathrm{Gal}_\kappa \times \mathrm{Gal}_\kappa \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X' \otimes \mathrm{Gal}_\kappa \arrow[r, shift left=1] \arrow[r, shift right=1] & X' \end{tikzcd}$$

compatibly with the analogous presentation of the Čech nerve $x_\bullet$ of $\bar{x} = \mathrm{Spec}(\overline{\kappa}) \to \mathrm{Spec}(\kappa) = x$. Applying $\Pi_\infty^{\mathrm{cond}}$ to the corresponding "ladder" diagram (coming from the map $x_\bullet \to X_\bullet$) and using that, for every $m \in \mathbf{N}$,

$$\mathrm{Gal}_\kappa^m \simeq \Pi_\infty^{\mathrm{cond}}(\bar{x} \otimes \mathrm{Gal}_\kappa^m) \to \Pi_\infty^{\mathrm{cond}}(X' \otimes \mathrm{Gal}_\kappa^m) \simeq \mathrm{Gal}_\kappa^m$$

is an isomorphism (where we are using Corollary 3.43 and the fact that both $\bar{x}$ and X' are connected w-contractible schemes), we conclude. ☐

4 Connected components of the condensed homotopy type

Let X be a qcqs scheme. In this section, we give an explicit description of the condensed set of connected components $\pi_0^{\mathrm{cond}}(X)$ of the condensed homotopy type $\Pi_\infty^{\mathrm{cond}}(X)$. To do so, we make use of the Galois category $\mathrm{Gal}(X_{\mathrm{zar}})$ of the Zariski ∞-topos in the sense of Definition 3.23. In §4.1, we show that the condensed connected components of $\mathrm{B}^{\mathrm{cond}}\mathrm{Gal}(X_{\mathrm{zar}})$ agree with $\pi_0^{\mathrm{cond}}(X)$. In §4.2, we use this description to show that if X has finitely many irreducible components, then $\pi_0^{\mathrm{cond}}(X)$ agrees with the profinite set $\pi_0(X)$ of connected components (Corollary 4.19). We also give examples of connected schemes whose $\pi_0^{\mathrm{cond}}(X)$ is nontrivial and show that $\pi_0^{\mathrm{cond}}(X)$ can be quite exotic in general. Finally, in §4.3, we use our explicit description of $\pi_0^{\mathrm{cond}}(X)$ to compute the condensed and étale homotopy types of the ring of continuous functions from a compact Hausdorff space to $\mathbf{C}$, see Corollary 4.33.

4.1 Pro-Zariski sheaves

Recall that for a scheme X, we will write X_{zar} for the ∞-topos of Zariski-sheaves on X.

4.1 Definition. Let X be a qcqs scheme. Let us write $X_{\mathrm{zar}}^{\mathrm{constr}}$ for the full subcategory of Zariski sheaves, that is spanned by the *constructible* sheaves on X, i.e., those sheaves that are constant with finite stalks on a finite constructible stratification of X. We call the ∞-topos

$$X_{\mathrm{prozar}}^{\mathrm{hyp}} := \mathrm{Sh}_{\mathrm{eff}}^{\mathrm{hyp}}(\mathrm{Pro}(X_{\mathrm{zar}}^{\mathrm{constr}}))$$

of hypersheaves for the effective epimorphism topology on $\mathrm{Pro}(X_{\mathrm{zar}}^{\mathrm{constr}})$, the hypercomplete *prozariski topos* of X. Since pullbacks along qcqs morphisms of schemes preserve constructible sheaves, $X_{\mathrm{prozar}}^{\mathrm{hyp}}$ is functorial in X.

4.2 Remark. This construction makes sense more generally for any *bounded coherent* ∞-topos (in the sense of [SAG, Appendix A]) and was called *solidification* in [9] and *pyknotification* in [80].

4.3. Let X be a qcqs scheme. The embedding $X_{\mathrm{zar}} \to X_{\text{ét}}$ preserves constructible sheaves and thus defines a functor

$$X_{\mathrm{zar}}^{\mathrm{constr}} \to X_{\text{ét}}^{\mathrm{constr}} .$$

Extending to proobjects we obtain a morphism of sites $\rho^* : \ \mathrm{Pro}(X_{\mathrm{zar}}^{\mathrm{constr}}) \to \mathrm{Pro}(X_{\text{ét}}^{\mathrm{constr}})$ and thus an algebraic morphism of ∞-topoi

$$X_{\mathrm{prozar}}^{\mathrm{hyp}} \to \mathrm{Sh}_{\mathrm{eff}}^{\mathrm{hyp}}(\mathrm{Pro}(X_{\text{ét}}^{\mathrm{constr}})) .$$

Finally, [55, Example 7.1.7] provides an equivalence $X_{\text{proét}}^{\mathrm{hyp}} \simeq \mathrm{Sh}_{\mathrm{eff}}^{\mathrm{hyp}}(\mathrm{Pro}(X_{\text{ét}}^{\mathrm{constr}}))$ so that we obtain an algebraic morphism

$$\rho^* : X_{\mathrm{prozar}}^{\mathrm{hyp}} \to X_{\text{proét}}^{\mathrm{hyp}} .$$

Recall that a map $Y \to X$ is a *Zariski localization* if Y is isomorphic (over X) to a finite disjoint union of open subschemes of X.

4.4. Let $X = \mathrm{Spec}(R)$ be affine scheme. We write $\mathrm{Zar}_X^{\mathrm{aff}}$ for the category of affine zariski localizations of X. Since open immersions between qcqs schemes are of finite presentation it follows from [STK, Tag 01ZC] that the canonical functor

$$\mathrm{Pro}(\mathrm{Zar}_X^{\mathrm{aff}}) \to \mathbf{Sch}_{/X}$$

is fully faithful and thus we may equip $\mathrm{Pro}(\mathrm{Zar}_X^{\mathrm{aff}})$ with the fpqc-topology. Since the sheaf represented by a Zariski localization is constructible, we obtain a morphism of sites

$$\mu : \ \mathrm{Pro}(\mathrm{Zar}_X^{\mathrm{aff}}) \to \mathrm{Pro}(X_{\mathrm{zar}}^{\mathrm{constr}})$$

4.5 Lemma. *Let X be an affine scheme. Then μ induces an equivalence of ∞-topoi*

$$\mathrm{Sh}_{\mathrm{fpqc}}^{\mathrm{hyp}}(\mathrm{Pro}(\mathrm{Zar}_X^{\mathrm{aff}})) \simeq X_{\mathrm{prozar}}^{\mathrm{hyp}}$$

Proof. The proof is exactly the same as in [55, Example 7.1.7].

□

4.6 Remark. Let X be an affine scheme. Then under the equivalence of Lemma 4.5, the functor ρ^* is induced by the morphism of sites

$$\mathrm{Pro}(\mathrm{Zar}_X^{\mathrm{aff}}) \to \mathrm{Pro}(\mathrm{\acute{E}t}_X^{\mathrm{aff}}) ,$$

that comes from the inclusion $\mathrm{Zar}_X^{\mathrm{aff}} \hookrightarrow \mathrm{\acute{E}t}_X^{\mathrm{aff}}$. Here $\mathrm{\acute{E}t}_X^{\mathrm{aff}}$ denotes the category of affine étale X-schemes.

4.7 Recollection. For a qcqs scheme X, we write $\mathrm{Gal}(X_{\mathrm{zar}})$ for the Galois category of the Zariski ∞-topos in the sense of Definition 3.23. Note that X_{zar} is the ∞-topos of sheaves on the spectral topological space $|X|$. Hence by Recollection 3.29, for a profinite set S, the category of sections $\mathrm{Gal}(X_{\mathrm{zar}})(S)$ is the poset of continuous quasicompact maps $f : S \to |X|$ ordered by pointwise specialization: $f \leq g$ if and only if for all $s \in S$, we have $f(s) \in \overline{\{g(s)\}}$. In particular, $\mathrm{Gal}(X_{\mathrm{zar}})(*)$ is the *specialization poset* of $|X|$, that we denote by $X_{\mathrm{zar}}^{\leq}$.

4.8 Lemma. *Let X be a qcqs scheme. Then there is a natural equivalence of ∞-topoi*

$$X_{\mathrm{prozar}}^{\mathrm{hyp}} \simeq \mathrm{Fun}^{\mathrm{cts}}(\mathrm{Gal}(X_{\mathrm{zar}}), \mathbf{Cond}(\mathbf{Ani})) .$$

Proof. Since X_{zar} is a spectral ∞-topos, in the sense of [8, Definition 9.2.1] with profinite stratified shape given by $\mathrm{Gal}(X_{\mathrm{zar}})$, this follows from [80, Theorem 1.1]. □

We are interested in the above result because it allows us to compute π_0 of the condensed homotopy type of the pro-Zariski ∞-topos as the condensed classifying anima of $\mathrm{Gal}(X_{\mathrm{zar}})$. The latter will be a quotient of the condensed set underlying $\mathrm{Gal}(X_{\mathrm{zar}})$ by an explicit equivalence relation. Furthermore, the next proposition will readily imply that this actually computes $\pi_0^{\mathrm{cond}}(X)$:

4.9 Proposition. *The functor $\rho^* : X_{\mathrm{prozar},\leq 0} \to X_{\mathrm{pro\acute{e}t},\leq 0}$ is fully faithful.*

In order to prove Proposition 4.9, we make use of the following construction:

4.10 Construction. Let $X = \mathrm{Spec}(R)$ be an affine scheme. Since the inclusion $\mathrm{Zar}_X^{\mathrm{aff}} \hookrightarrow \mathrm{\acute{E}t}_X^{\mathrm{aff}}$ preserves finite limits, it admits a pro left adjoint

$$\mathrm{Hens}_{\mathrm{X}}^{\mathrm{zar}} : \mathrm{Pro}(\mathrm{\acute{E}t}_X^{\mathrm{aff}}) \to \mathrm{Pro}(\mathrm{Zar}_X^{\mathrm{aff}}) .$$

4.11 Definition (Zariski henselization)**.** Let $X = \mathrm{Spec}(R)$ be an affine scheme. Given any $Y \in \mathrm{Pro}(\mathrm{\acute{E}t}_X^{\mathrm{aff}})$, we call $\mathrm{Hens}_{\mathrm{X}}^{\mathrm{zar}}(Y)$ the *Zariski henselization of Y in X*.

4.12 Lemma. *Let $V \in \mathrm{Pro}(\mathrm{\acute{E}t}_X^{\mathrm{aff}})$. If V is w-contractible, the unit morphism*

$$V \to \mathrm{Hens}_{\mathrm{X}}^{\mathrm{zar}}(V)$$

is surjective.

Proof. Since V is w-contractible, we can use the universal property of $\mathrm{Hens}_{\mathrm{X}}^{\mathrm{zar}}(V)$ to show that any pro-Zariski cover of $\mathrm{Hens}_{\mathrm{X}}^{\mathrm{zar}}(V)$ has a section. This in particular shows that $\mathrm{Hens}_{\mathrm{X}}^{\mathrm{zar}}(V)$ is w-local, see [10, Lemma 2.4.2]. Since $V \to \mathrm{Hens}_{\mathrm{X}}^{\mathrm{zar}}(V)$ is flat and the image of a flat morphism is closed under generization [30, Lemma 14.9], it suffices to see that all closed points are in the image. We now assume that $\mathrm{im}(V) \subset \mathrm{Hens}_{\mathrm{X}}^{\mathrm{zar}}(V)$ does not contain a closed point x. Since $\mathrm{im}(V)$ is quasicompact, there is some quasicompact open $H \subset \mathrm{Hens}_{\mathrm{X}}^{\mathrm{zar}}(V)$ containing $\mathrm{im}(V)$ while $x \notin H$.

Since H is quasicompact, we may find a covering by finitely many affines $U_i = \mathrm{Spec}(R_i) \to H$. Since $\mathrm{im}(V) \subset H$, it follows that the induced map

$$\coprod_i U_i \times_{\mathrm{Hens}^{\mathrm{zar}}_{\mathrm{X}}(V)} V \to V$$

is surjective and thus admits a section $\alpha \colon V \to \coprod_i U_i \times_{\mathrm{Hens}^{\mathrm{zar}}_{\mathrm{X}}(V)} V$. By the universal property of Zariski henselization, the composition

$$V \xrightarrow{\alpha} \textstyle\coprod_i U_i \times_{\mathrm{Hens}^{\mathrm{zar}}_{\mathrm{X}}(V)} V \longrightarrow \textstyle\coprod_i U_i$$

factors uniquely through some $\tilde{\alpha} \colon \mathrm{Hens}^{\mathrm{zar}}_{\mathrm{X}}(V) \to \coprod_i U_i$. Since the composite

$$V \xrightarrow{\alpha} \textstyle\coprod_i U_i \times_{\mathrm{Hens}^{\mathrm{zar}}_{\mathrm{X}}(V)} V \longrightarrow \textstyle\coprod_i U_i \longrightarrow \mathrm{Hens}^{\mathrm{zar}}_{\mathrm{X}}(V)$$

recovers the unit $V \to \mathrm{Hens}^{\mathrm{zar}}_{\mathrm{X}}(V)$, it follows by uniqueness that the composite

$$\mathrm{Hens}^{\mathrm{zar}}_{\mathrm{X}}(V) \xrightarrow{\tilde{\alpha}} \textstyle\coprod_i U_i \longrightarrow \mathrm{Hens}^{\mathrm{zar}}_{\mathrm{X}}(V)$$

is the identity. In particular the U_i cover $\mathrm{Hens}^{\mathrm{zar}}_{\mathrm{X}}(V)$ and thus $H = \mathrm{Hens}^{\mathrm{zar}}_{\mathrm{X}}(V)$, which contradicts that $x \notin H$. □

4.13 Lemma. *Let X be an affine scheme, and $F \in X^{\mathrm{hyp}}_{\mathrm{prozar}}$. Then ρ^*F is the sheafification of*

$$\mathrm{Pro}(\text{Ét}^{\mathrm{aff}}_X) \ni W \mapsto F(\mathrm{Hens}^{\mathrm{zar}}_{\mathrm{X}}(W)).$$

Moreover, if W is w-contractible, then $\rho^*F(W) = F(\mathrm{Hens}^{\mathrm{zar}}_{\mathrm{X}}(W))$.

Proof. The functor ρ^* is given by sheafification of the left Kan extension along the map

$$\iota \colon \mathrm{Pro}(\mathrm{Zar}^{\mathrm{aff}}_X) \hookrightarrow \mathrm{Pro}(\text{Ét}^{\mathrm{aff}}_X)\,.$$

Explicitly, for $F \in X^{\mathrm{hyp}}_{\mathrm{prozar}}$ the image is given as

$$\rho^*F = (W \mapsto \operatorname*{colim}_{W \to \iota(V)} F(V))^{\#}$$

for $V \in \mathrm{Pro}(\mathrm{Zar}^{\mathrm{aff}}_X)$ and $U \in \mathrm{Pro}(\text{Ét}^{\mathrm{aff}}_X)$. By the universal property of $\mathrm{Hens}^{\mathrm{zar}}_{\mathrm{X}}$, every map $W \to \iota(V)$ factors uniquely over $\mathrm{Hens}^{\mathrm{zar}}_{\mathrm{X}}(W) \in \mathrm{Pro}(\mathrm{Zar}^{\mathrm{aff}}_X)$, hence the colimit above reduces to

$$\operatorname*{colim}_{W \to \iota(V)} F(V) = F(\mathrm{Hens}^{\mathrm{zar}}_{\mathrm{X}}(W)).$$

It remains to argue why sheafification can be omitted for w-contractible W. On the basis of w-contractible affines weakly étale over X, the sheaf condition simplifies to preservation of finite products. Since $\mathrm{Hens}^{\mathrm{zar}}_{\mathrm{X}}$, being a left adjoint, preserves finite coproducts and F carries such to finite products, the claim follows. □

Proof of Proposition 4.9. We can immediately reduce to the case where X is affine. We want to show that for any $F \in X_{\mathrm{prozar},\le 0}$ and any $U \in \mathrm{Pro}(\mathrm{Zar}_X^{\mathrm{aff}})$ the unit evaluated at U

$$F(U) \to \rho^*(F)(U)$$

is an isomorphism. For this, pick a w-contractible weakly étale X-scheme W with a surjection $W \twoheadrightarrow U$ and a further w-contractible V with a surjection $V \twoheadrightarrow W \times_U W$. Using Lemma 4.13, it suffices to see that the canonical map

$$F(U) \to \lim \left(F(\mathrm{Hens}_X^{\mathrm{zar}}(W)) \rightrightarrows F(\mathrm{Hens}_X^{\mathrm{zar}}(V)) \right)$$

is an isomorphism. This is clear if we show that

$$\mathrm{Hens}_X^{\mathrm{zar}}(V) \rightrightarrows \mathrm{Hens}_X^{\mathrm{zar}}(W) \to U$$

is the beginning of an augmented pro-Zariski hypercover. For this, first observe that since the surjection $W \twoheadrightarrow U$ factors through the canonical map $\mathrm{Hens}_X^{\mathrm{zar}}(W) \to U$, the rightmost morphism above is surjective. Note that we have a commutative diagram

$$\begin{tikzcd}
V \arrow[r, two heads] \arrow[d] & W \times_U W \arrow[d] \\
\mathrm{Hens}_X^{\mathrm{zar}}(V) \arrow[r] & \mathrm{Hens}_X^{\mathrm{zar}}(W) \times_U \mathrm{Hens}_X^{\mathrm{zar}}(W).
\end{tikzcd}$$

Here the top horizontal morphism is surjective by definition and the right vertical morphism is surjective by Lemma 4.12. Thus the bottom horizontal morphism is also surjective, as desired. □

4.14 Remark. Note that Proposition 4.9 is only true on the level of 0-truncated sheaves, i.e., sheaves of sets. Full faithfulness on the level of sheaves of anima would imply an equivalence of the condensed homotopy type and the pro-Zariski shape (relative to condensed anima). Therefore, it would also imply that the étale homotopy type of X agrees with the shape of the underlying topological space of X, which is clearly false. However, it is true for everywhere strictly local schemes X as one has $X_{\text{ét}} = X_{\mathrm{zar}}$ by [71, Corollary 2.5].

4.2 An explicit description of π_0^{cond}

Together the results from the last section show:

4.15 Proposition. *Let X be a qcqs scheme. Then there is a natural isomorphism of condensed sets*

$$\pi_0^{\mathrm{cond}}(X) \xrightarrow{\sim} \pi_0(\mathrm{B}^{\mathrm{cond}}\mathrm{Gal}(X_{\mathrm{zar}}))\,.$$

Proof. We have a commutative triangle

$$\begin{tikzcd}
\mathrm{Cond}(\mathbf{Ani}) \arrow[r, "\tilde{\pi}^*"] \arrow[dr, "\pi^*"'] & X_{\mathrm{prozar}}^{\mathrm{hyp}} \arrow[d, "\rho^*"] \\
& X_{\mathrm{pro\acute{e}t}}^{\mathrm{hyp}}
\end{tikzcd}$$

where $\tilde{\pi}$ is induced by the morphism of sites

$$\mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}}) \to \mathrm{Pro}(X_{\mathrm{zar}}^{\mathrm{constr}}); \quad S \mapsto S \times X.$$

Combining Lemma 4.8 and [80, Lemma 4.3], it follows that $\tilde{\pi}^*$ has a left adjoint, that we denote $\tilde{\pi}_\sharp$. By Proposition 4.9, it follows that $\pi_0^{\mathrm{cond}}(X) \simeq \pi_0(\tilde{\pi}_\sharp(1))$. Then by Lemma 4.8, we can use the same argument as in Proposition 3.36 to show that $\tilde{\pi}_\sharp(1) \simeq \mathrm{B}^{\mathrm{cond}}\mathrm{Gal}(X_{\mathrm{zar}})$, as desired. ☐

Thus, we may quite explicitly describe $\pi_0^{\mathrm{cond}}(X)$.

4.16 Remark. The next theorem involves sets of continuous quasicompact maps $\mathrm{Map}_{\mathrm{qc}}(S, T)$ where S is a profinite set and T is a spectral space. Note that these are precisely those maps such that the preimage of a quasicompact open is clopen. It then follows that these are precisely continuous maps in the constuctible topology, i.e.,

$$\mathrm{Map}_{\mathrm{qc}}(S, T) = \mathrm{Map}(S, T^{\mathrm{constr}})\,.$$

Said differently, the inclusion of the full subcategory of profinite sets into the category of spectral spaces and quasicompact maps admits a right adjoint, given by sending a spectral space T to the underlying set of T equipped with the constructible topology.

4.17 Theorem. *Let X be a qcqs scheme. Then for any extremally disconnected profinite set S, we have*

$$\pi_0^{\mathrm{cond}}(X)(S) = \mathrm{Map}_{\mathrm{qc}}(S, |X|)/\sim\ ,$$

where $f \sim g$ if and only if there is some $n \in \mathbf{N}$ and quasicompact maps $s_1, t_1, \dots, s_n, t_n : S \to |X|$ such that

$$f \geq s_1 \leq t_1 \geq s_2 \leq t_2 \geq \cdots \geq s_n \leq t_n \geq g\,,$$

where $a \leq b$ if and only if $a(s) \in \overline{\{b(s)\}}$ for all $s \in S$. If $S = \beta(M)$ for some discrete set M, we furthermore have a canonical isomorphism

$$\pi_0^{\mathrm{cond}}(X)(\beta(M)) \simeq \pi_0((X_{\mathrm{zar}}^{\leq})^M)\,.$$

Proof. By Proposition 4.15, the first part of the theorem reduces to showing that for every extremally disconnected profinite set S, we have

$$\pi_0(\mathrm{B}^{\mathrm{cond}}\mathrm{Gal}(X_{\mathrm{zar}}))(S) = \mathrm{Map}_{\mathrm{qc}}(S, |X|)/\sim\ .$$

This follows by the description of $\mathrm{Gal}(X_{\mathrm{zar}})$ in Recollection 4.7 noticing that two maps f, g in the poset $\mathrm{Map}_{\mathrm{qc}}(S, |X|)$ are connected if and only if there exists a finite zig-zag of specializations as indicated in the statement. If $S = \beta(M)$ for some discrete set M, we have by Proposition 2.22

$$\begin{aligned} \mathrm{Map}_{\mathrm{qc}}(\beta(M), |X|) &\simeq \mathrm{Gal}(X_{\mathrm{zar}})(\beta(M)) \\ &\simeq \prod_M \mathrm{Gal}(X_{\mathrm{zar}})(*) \simeq \prod_M X_{\mathrm{zar}}^{\leq}\,. \end{aligned}$$ ☐

4.18 Construction. Let X be a qcqs scheme. The image of the condensed connected components $\pi_0^{\mathrm{cond}}(X)$ under the left adjoint $(-)^{\wedge}_{\mathrm{disc}} : \mathrm{Cond}(\mathbf{Ani}) \to \mathrm{Pro}(\mathbf{Ani}_{<\infty})$ coincides with the profinite set of connected components $\pi_0(X) \in \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}}) \subset \mathrm{Pro}(\mathbf{Ani})$ after 0-truncation. Indeed, by Lemma 3.14, the above is given by the connected components of $\Pi_{<\infty}^{\mathrm{ét}}(X)$. Thus, the 0-truncation of the unit $\Pi_\infty^{\mathrm{cond}}(X) \to \Pi_{<\infty}^{\mathrm{ét}}(X) \in \mathrm{Cond}(\mathbf{Ani})$ gives a natural map of condensed sets

$$\varphi : \pi_0^{\mathrm{cond}}(X) \to \pi_0(X).$$

Theorem 4.17 shows that $\pi_0^{\mathrm{cond}}(X)$ gives the expected answer in some cases:

4.19 Corollary. *Let X be a qcqs scheme with finitely many irreducible components. Then the natural map of condensed sets*

$$\varphi\colon \pi_0^{\mathrm{cond}}(X) \to \pi_0(X)$$

is an isomorphism.

Proof. It suffices to check that φ is an isomorphism after evaluating at $\beta(M)$ for any discrete set M. By Theorem 4.17, we need to see that the canonical map

$$\pi_0((X_{\mathrm{zar}}^{\leq})^M) \to \pi_0(X)^M$$

that sends a function $M \to |X|$ to the composite with $|X| \to \pi_0(X)$ is an isomorphism (note that this is not immediate, since in general π_0 does not commute with infinite products). It is surjective by surjectivity of $|X| \to \pi_0(X)$. For injectivity, suppose that we have two maps $f, g\colon M \to |X|$ that agree after composing with π_0. If the number of irreducible components of X is n, it follows that we may connect any two points $x, y \in X$ in the same connected component with a zig-zag of specializations involving at most $2n+1$ other points. Thus we may also connect f and g with a zig-zag involving $2n+1$ other maps and thus $[f] = [g]$ in $\pi_0((X_{\mathrm{zar}}^{\leq})^M)$, as desired. ☐

4.20 Remark. For an alternative proof of Corollary 4.19, see [56, Proposition 2.2.25].

4.21 Recollection [24, Chapter 0, §2.3]**.** A spectral space T is *valuative* if, for each $t \in T$, the set of generizations of t is totally ordered under the generization relation. Every point t of a valuative space T has a unique maximal generization, denoted $t^{\max}$.

The *separated quotient* of a valuative spectral space T is the quotient $T^{\mathrm{sep}} \coloneqq T/\sim$ by the relation $s \sim t$ if $s^{\max} \sim t^{\max}$. By [24, Chapter 0, Corollary 2.3.18], T^{sep} is a compact Hausdorff space.

For the next result, recall the Galois category of a spectral space from Notation 3.28 and Recollection 3.29.

4.22 Corollary. *Let T be a valuative spectral space. Then the natural map*

$$\pi_0(\mathrm{Gal}(T_{\mathrm{zar}})) \to T^{\mathrm{sep}}$$

is an isomorphism of condensed sets.

Proof. It again suffices to check this after evaluating at any $\beta(M)$. So let $\alpha\colon \beta(M) \to T^{\mathrm{sep}}$ be any continuous map. Since the quotient map $\pi\colon T \to T^{\mathrm{sep}}$ is surjective, we may pick a map $a\colon M \to T$ lifting $\alpha|_M$. Using Proposition 2.22 as in Theorem 4.17, a extends to a quasicompact continuous map $\bar{a}\colon \beta(M) \to T$ and by construction we have $\pi \circ \bar{a}|_M = \alpha|_M$. By the universal property of Čech–Stone compactification, we thus get $\pi \circ \bar{a} = \alpha$, proving surjectivity. For injectivity, suppose that we are given two maps $f, g\colon M \to T$ such that the composites with π agree. By the valuative property, it follows that for any $m \in M$, $f(m)$ and $g(m)$ specialize to the same maximal element $h(m)$. Thus we get a zig-zag

$$f \leq h \geq g$$

so that $[f] = [g]$ in $\pi_0(\mathrm{Gal}(T_{\mathrm{zar}}))(\beta(M))$, proving injectivity. ☐

4.23 Example. Corollary 4.22 shows that even if X is a connected scheme, $\pi_0^{\mathrm{cond}}(X)$ can be a nontrivial condensed set. Concretely, we may take T to be the underlying topological space of the adic unit disk. Then T is a connected spectral topological space, so there exists a ring R and a homeomorphism $T \simeq |\mathrm{Spec}(R)|$. Thus $\mathrm{Spec}(R)$ is connected but $\pi_0^{\mathrm{cond}}(\mathrm{Spec}(R)) = T^{\mathrm{sep}}$ is a nontrivial compact Hausdorff space. In fact, this space is homeomorphic to the underlying space of the corresponding Berkovich disk (cf. [44, Remark 8.3.2]).

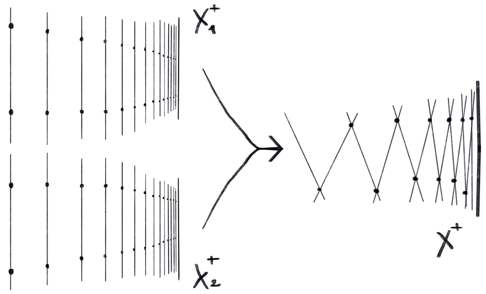
Theorem 4.17 can also be used to show that for a general qcqs scheme X, the condensed set $\pi_0^{\mathrm{cond}}(X)$ can be quite exotic (in particular, $\pi_0^{\mathrm{cond}}(X)$ is not generally quasiseparated in the sense of Recollection 7.7). This is achieved in the following example.

4.24 Example (schematic Warsaw circle)**.** Let X be a qcqs scheme with the property that any two points may be connected by a zig-zag of specializations but such that the minimal length of such a chain is not bounded by any natural number. Then we have

$$\pi_0^{\mathrm{cond}}(X)(*) \simeq * \,.$$

However, for any function $f : \mathbf{N} \to |X|$ such that the minimal length of a zig-zag connecting $f(n)$ and $f(0)$ is at least n, the function f and the constant function at $f(0)$ yield different elements in $\pi_0^{\mathrm{cond}}(X)(\beta(\mathbf{N}))$. Thus, $\pi_0^{\mathrm{cond}}(X)$ is a nontrivial condensed set whose underlying set is the point and therefore not quasiseparated. Indeed, if it was quasiseparated it would be qcqs and thus representable by a compact Hausdorff space.

Let us give a concrete example of a scheme satisfying these properties. Fix an algebraically closed field $k = \bar{k}$ and write $* = \mathrm{Spec}(k)$. Let $X \in *_{\text{proét}}$ be a scheme such that $\pi_0(X) = \mathbf{N} \cup \infty$, i.e., the converging sequence of points together with its limit. Each connected component of X is just a copy of $*$. Take two copies $X_1^+ = X_2^+ = \mathbf{A}_k^1 \times_{\mathrm{pt}} X$ of a scheme that, intuitively, is a sequence of affine lines converging to another affine line. Fix two points, say $0, 1$, on each copy of $\mathbf{A}_k^1$ and glue X_1^+ and X_2^+ to obtain a *zigzag* of $\mathbf{A}_k^1$'s intersecting at 0's and 1's and converging to a copy of $\mathbf{A}_k^1$. Picture:



Let us denote this scheme simply by X^+. To formalize this gluing procedure, one notes that we are gluing affine schemes along closed subschemes, and so the pushout exists (and is an affine scheme again) by [72, Theorem 3.4].

Now, this scheme satisfies the condition of having specialization-distances between points growing arbitrarily but it still needs a small correction: the points on the limit $\mathbf{A}_k^1$ are not joinable by a specialization sequence with the points on the zigzag. To amend it, add a further copy of $\mathbf{A}_k^1$ joining an arbitrarily chosen pair of k-points of the the leftmost line of the zigzag with the limit line of X^+. Let us denote by X^{++} this schematic 'Warsaw circle'. One can check that X^{++} satisfies the desired properties.

4.3 Computation: $\Pi_\infty^{\mathrm{cond}}$ of rings of continuous functions

Let T be a compact Hausdorff space. We conclude this section by using Theorem 4.17 to compute the condensed homotopy type of the ring of continuous functions $\mathrm{C}(T, \mathbf{C})$; we show that it is 0-truncated, and coincides with the condensed set represented by T. We accomplish this by proving a more general result. To state it, recall that the ring $\mathrm{C}(T, \mathbf{C})$ has the property that every prime ideal is contained in a unique maximal ideal (see Theorem A.24). Moreover, [67, Chapitre VII, Proposition 4] shows that the local rings of $\mathrm{C}(T, \mathbf{C})$ at maximal ideals are strictly henselian. We are able to compute the condensed homotopy types of rings satisfying these two properties.

To state our results, we first introduce some terminology.

4.25 Notation. Given a ring R, we write $\mathrm{MSpec}(R) \subset |\mathrm{Spec}(R)|$ for the subset of maximal ideals, endowed with the subspace topology.

4.26 Recollection (see Appendix A)**.** A ring R is a *pm-ring* if every prime ideal of R is contained in a unique maximal ideal. In this case, the space $\mathrm{MSpec}(R)$ is compact Hausdorff.

First, we identify π_0^{cond} of an arbitrary pm-ring.

4.27 Proposition. *Let R be a pm-ring. Then there is a natural isomorphism of condensed sets*

$$\pi_0^{\mathrm{cond}}(\mathrm{Spec}(R)) \xrightarrow{\sim} \mathrm{MSpec}(R)\,.$$

This isomorphism is constructed in the course of the proof.

Proof. By Theorem A.9, the map of topological spaces $|\mathrm{Spec}(R)| \to \mathrm{MSpec}(R)$ that sends a prime ideal $\mathfrak{p}$ to the unique maximal ideal containing $\mathfrak{p}$ is a continuous retraction of the inclusion. This retraction is also continuous for the constructible topology and therefore defines a map of condensed sets

$$\mathrm{Map}_{\mathbf{Top}}(-, |\mathrm{Spec}(R)|^{\mathrm{cons}}) \to \mathrm{MSpec}(R)\,.$$

Furthermore it clearly respects the equivalence relation described in Theorem 4.17 and therefore induces a map

$$\pi_0^{\mathrm{cond}}(\mathrm{Spec}(R)) \to \mathrm{MSpec}(R)\,.$$

To check that this map is an isomorphism, it suffices to check this after evaluating at $\beta(M)$ for any set M. Using the explicit description given in Theorem 4.17 and the fact that $\mathrm{MSpec}(R)$ is compact Hausdorff (Corollary A.10), this is immediate. ☐

Under stronger hypotheses, we compute the whole condensed homotopy type:

4.28 Theorem. *Let R be a pm-ring with the property that all local rings at maximal ideals are strictly henselian. Then $\Pi_\infty^{\mathrm{cond}}(\mathrm{Spec}(R))$ is 0-truncated; hence there is a natural equivalence of condensed anima*

$$\Pi_\infty^{\mathrm{cond}}(\mathrm{Spec}(R)) \xrightarrow{\sim} \mathrm{MSpec}(R)\,.$$

To show that $\Pi_\infty^{\mathrm{cond}}(\mathrm{Spec}(R))$ is 0-truncated, we use the description of the condensed homotopy type via exodromy. We first prove some preparatory results about classifying anima of infinite products.

4.29 Lemma. *Let I be a set and let $(\mathcal{C}_i)_{i \in I}$ be ∞-categories. Assume that for each $i \in I$, there exists a left adjoint functor $\lambda_i \colon A_i \to \mathcal{C}_i$ where A_i is an anima. Then all of the maps in the commutative square*

$$\begin{tikzcd}
\mathrm{B}(\prod_{i \in I} A_i) \arrow[r] \arrow[d, "\mathrm{B}(\prod_{i \in I} \lambda_i)"'] & \prod_{i \in I} \mathrm{B} A_i \arrow[d, "\prod_{i \in I} \mathrm{B}\lambda_i"] \\
\mathrm{B}(\prod_{i \in I} \mathcal{C}_i) \arrow[r] & \prod_{i \in I} \mathrm{B}\mathcal{C}_i \, .
\end{tikzcd}$$

are equivalences of anima.

Proof. First observe that since each λ_i is a left adjoint, the induced functor on products

$$\prod_{i \in I} \lambda_i \colon \prod_{i \in I} A_i \to \prod_{i \in I} \mathcal{C}_i$$

is also a left adjoint. Since each A_i is an anima, the top horizontal map is an equivalence. Since $\prod_{i \in I} \lambda_i$ and each λ_i is a left adjoint and the functor $\mathrm{B} \colon \mathbf{Cat}_\infty \to \mathbf{Ani}$ sends left adjoints to equivalences [15, Corollary 2.11], the vertical maps are also equivalences. Thus, by the 2-of-3 property, the bottom horizontal map is an equivalence, as desired. ☐

4.30 Example. Let I be a set and let $(\mathcal{C}_i)_{i \in I}$ be ∞-categories. Assume that for each $i \in I$, each connected component of the ∞-category $\mathcal{C}_i$ admits an initial object. Then the hypotheses of Lemma 4.29 are satisfied where each A_i is the set of initial objects of connected components of $\mathcal{C}_i$ and λ_i is the inclusion. In particular,

$$\mathrm{B}(\prod_{i \in I} \mathcal{C}_i) \simeq \prod_{i \in I} \mathrm{B}\mathcal{C}_i$$

is 0-truncated.

We also need the following criterion for detecting when a condensed anima is 0-truncated:

4.31 Lemma. *Let $n \geq 0$ be an integer. Then a condensed anima A is n-truncated if and only if for each set M, the anima $A(\beta(M))$ is n-truncated.*

Proof. Since every extremally disconnected profinite set is a retract of the Čech–Stone compactification of a set, this follows from the fact that every retract of an n-truncated anima is n-truncated. ☐

Proof of Theorem 4.28. Note that, in light of Proposition 4.27, the final statement follows from the claim that $\Pi_\infty^{\mathrm{cond}}(\mathrm{Spec}(R))$ is 0-truncated; so we just show this. Let us write $X = \mathrm{Spec}(R)$. By Lemma 4.31, it suffices to show that for every set M, the classifying anima of the category $\mathrm{Gal}(X)(\beta(M))$ is 0-truncated. Together, Recollection 3.31 and Proposition 2.22 show that

$$\mathrm{Gal}(X)(\beta(M)) \simeq \prod_{m \in M} \mathrm{Gal}(X)(\{m\}) \simeq \prod_{m \in M} \mathrm{Pt}(X_{\mathrm{\acute{e}t}}) \, .$$

So by Example 4.30, it suffices to show that every connected component of $\mathrm{Pt}(X_{\mathrm{\acute{e}t}})$ has an initial object. This last statement is immediate from the assumption that R is a pm-ring and all local rings at maximal ideals are strictly henselian. ☐

We now derive some consequences of Theorem 4.28. The first is a computation of the étale homotopy type of these pm-rings, which appears to be new.

4.32 Corollary. *Let R be a pm-ring with the property that all local rings at maximal ideals are strictly henselian. Then there is a canonical equivalence of proanima*

$$\Pi^{\text{ét}}_{<\infty}(\operatorname{Spec}(R)) \xrightarrow{\sim} \Pi_{<\infty}(\operatorname{MSpec}(R)) .$$

Here, $\Pi_{<\infty}(\operatorname{MSpec}(R))$ denotes the shape of the compact Hausdorff space $\operatorname{MSpec}(R)$. See Notation 2.24.

Proof. We apply the functor $(-)^{\wedge}_{\text{disc}} \colon \operatorname{Cond}(\mathbf{Ani}) \to \operatorname{Pro}(\mathbf{Ani}_{<\infty})$ to the equivalence in Theorem 4.28. To conclude, note that by Lemma 3.14, we have

$$\Pi^{\text{cond}}_{\infty}(\operatorname{Spec}(R))^{\wedge}_{\text{disc}} \simeq \Pi^{\text{ét}}_{<\infty}(\operatorname{Spec}(R))$$

and by Lemma 2.26 we have

$$\operatorname{MSpec}(R)^{\wedge}_{\text{disc}} \simeq \Pi_{<\infty}(\operatorname{MSpec}(R)) . \qquad \square$$

Finally, we turn to the special case of rings of continuous functions.

4.33 Corollary. *Let T be a topological space and let $\mathrm{C_b}(T, \mathbf{C})$ denote the ring of bounded continuous functions to $\mathbf{C}$. Then there are natural equivalences*

$$\Pi^{\text{cond}}_{\infty}(\operatorname{Spec}(\mathrm{C_b}(T, \mathbf{C}))) \xrightarrow{\sim} \beta(T)$$

and

$$\Pi^{\text{ét}}_{<\infty}(\operatorname{Spec}(\mathrm{C_b}(T, \mathbf{C}))) \xrightarrow{\sim} \Pi_{<\infty}(\beta(T)) .$$

4.34. Note that if T is compact Hausdorff, then $\beta(T) = T$ and $\mathrm{C_b}(T, \mathbf{C}) = \mathrm{C}(T, \mathbf{C})$.

Proof. By the universal property of Čech–Stone compactification, the natural map $T \to \beta(T)$ induces an isomorphism of rings

$$\mathrm{C}(\beta(T), \mathbf{C}) \xrightarrow{\sim} \mathrm{C_b}(T, \mathbf{C}) .$$

By Theorem A.24, the ring $\mathrm{C}(\beta(T), \mathbf{C})$ is a pm-ring and by Theorem A.32 there is a natural homeomorphism $\beta(T) \xrightarrow{\sim} \operatorname{MSpec}(\mathrm{C}(\beta(T), \mathbf{C}))$. Furthermore, [67, Chapitre VII, Proposition 4] shows that the local rings of $\mathrm{C}(\beta(T), \mathbf{C})$ at maximal ideals are strictly henselian. Thus the claim follows from Theorem 4.28 and Corollary 4.32 applied to $R = \mathrm{C}(\beta(T), \mathbf{C})$. $\square$

4.35 Remark. Let T be a compact Hausdorff space that admits a CW structure and $t \in T$. Since T admits a CW structure, the shape $\Pi_{\infty}(T)$ coincides with the underlying anima of T. Hence Corollary 4.33 shows that, up to protruncation, the étale homotopy type of $\operatorname{Spec}(\mathrm{C}(T, \mathbf{C}))$ coincides with the underlying anima of T. In particular, the SGA3 étale fundamental group of $\operatorname{Spec}(\mathrm{C}(T, \mathbf{C}))$ at the maximal ideal of functions vanishing at t coincides with the usual fundamental group $\pi_1(T, t)$.

5 Fiber sequences

Let k be a field with separable closure $\bar{k} \supset k$, and let X be a qcqs k-scheme. Write $X_{\bar{k}}$ for the basechange of X to $\bar{k}$. Then the naturally null sequence of étale homotopy types

$$\Pi^{\text{ét}}_{<\infty}(X_{\bar{k}}) \longrightarrow \Pi^{\text{ét}}_{<\infty}(X) \longrightarrow \mathrm{BGal}_k \tag{5.1}$$

is a fiber sequence, see [36, Theorem 0.2]. The existence of this fiber sequence implies the usual fundamental exact sequence for étale fundamental groups [STK, Tag 0BTX; SGA 1, Exposé IX, Théorème 6.1].

The first goal of this section, accomplished in § 5.1, is to prove the analogue of the fundamental fiber sequence (5.1) for the condensed homotopy type. The second goal of this section, accomplished in § 5.2, is to show that given a smooth proper morphism of schemes $X \to S$, up to suitable completion, the homotopy-theoretic fiber of the induced map $\Pi_\infty^{\mathrm{cond}}(X) \to \Pi_\infty^{\mathrm{cond}}(S)$ agrees with the condensed homotopy type of the scheme-theoretic fiber. See Theorem 5.12.

5.1 The fundamental fiber sequence for the condensed homotopy type

Using the description of $\Pi_\infty^{\mathrm{cond}}(X)$ as the condensed classifying anima $\mathrm{B}^{\mathrm{cond}}\mathrm{Gal}(X)$, the same methods as in [36] allow us to prove the fundamental fiber sequence for the condensed homotopy type. The key observation is that even though $\mathrm{B}^{\mathrm{cond}}$ does not preserve pullbacks, it preserves pullbacks along morphisms between condensed anima. Let us now explain this point.

5.2 Recollection. Let $\mathcal{C}$ be an ∞-category with pullbacks and $\mathcal{D} \subset \mathcal{C}$ a full subcategory such that the inclusion admits a left adjoint $L \colon \mathcal{C} \to \mathcal{D}$. We say that the localization L is *locally cartesian* if for any cospan $U \to W \leftarrow V$ in $\mathcal{C}$ with $U, W \in \mathcal{D}$, the natural map

$$L(U \times_W V) \to U \times_W L(V)$$

is an equivalence. See [26, §1.2; 43, §3.2].

5.3. Importantly, the localization $\mathrm{B} \colon \mathbf{Cat}_\infty \to \mathbf{Ani}$ is locally cartesian; see [36, Example 3.4].

5.4 Corollary. *Let $\mathcal{C}$ be an ∞-category with finite limits and let $L \colon \mathcal{C} \to \mathcal{D}$ be a locally cartesian localization that also perserves finite products. Then the localization $L^{\mathrm{cond}} \colon \mathrm{Cond}(\mathcal{C}) \to \mathrm{Cond}(\mathcal{D})$ is locally cartesian.*

Proof. By definition, the functor

$$L^{\mathrm{cond}} \colon \mathrm{Fun}^{\times}(\mathbf{Extr}^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}^{\times}(\mathbf{Extr}^{\mathrm{op}}, \mathcal{D})$$

is given by pointwise application of $L \colon \mathcal{C} \to \mathcal{D}$. Since finite limits in $\mathrm{Cond}(\mathcal{C})$ and $\mathrm{Cond}(\mathcal{D})$ are computed pointwise, the claim follows from the assumption that the localization L is locally cartesian. ☐

5.5 Example. The localization $\mathrm{B}^{\mathrm{cond}} \colon \mathrm{Cond}(\mathbf{Cat}_\infty) \to \mathrm{Cond}(\mathbf{Ani})$ is locally cartesian.

5.6 Corollary. *Let $f \colon X \to S$ be a morphism between qcqs schemes, and let $\bar{s} \to S$ be a geometric point of S. If $\dim(S) = 0$, then the naturally null sequence*

$$\Pi_\infty^{\mathrm{cond}}(X_{\bar{s}}) \longrightarrow \Pi_\infty^{\mathrm{cond}}(X) \longrightarrow \Pi_\infty^{\mathrm{cond}}(S)$$

is a fiber sequence in the ∞-category $\mathrm{Cond}(\mathbf{Ani})$. As a consequence, given a geometric point $\bar{x} \to X_{\bar{s}}$, the induced sequence of pointed condensed sets

$$1 \longrightarrow \pi_1^{\mathrm{cond}}(X_{\bar{s}}, \bar{x}) \longrightarrow \pi_1^{\mathrm{cond}}(X, \bar{x}) \longrightarrow \pi_1^{\mathrm{cond}}(S, \bar{s}) \longrightarrow \pi_0^{\mathrm{cond}}(X_{\bar{s}}) \longrightarrow \pi_0^{\mathrm{cond}}(X) \longrightarrow \pi_0^{\mathrm{cond}}(S)$$

is exact.

Proof. For the first claim, note that by [36, Corollary 2.4] and the fact that the functor $\mathrm{Pro}(\mathbf{Cat}_\infty) \to \mathrm{Cond}(\mathbf{Cat}_\infty)$ preserves limits, the natural square

$$\begin{tikzcd}
\mathrm{Gal}(X_{\bar{s}}) \arrow[r] \arrow[d] & \mathrm{Gal}(X) \arrow[d] \\
\mathrm{Gal}(\bar{s}) \arrow[r] & \mathrm{Gal}(S)
\end{tikzcd}$$

is a pullback square in $\mathrm{Cond}(\mathbf{Cat}_\infty)$. Moreover, since $\bar{s}$ is a geometric point, $\mathrm{Gal}(\bar{s}) \simeq *$. Since $\dim(S) = 0$, by Corollary 3.37 the condensed ∞-category $\mathrm{Gal}(S)$ is a 1-truncated condensed anima. The claim now follows from Proposition 3.36 and the fact that the localization $\mathrm{B}^{\mathrm{cond}}$ is locally cartesian.

To conclude, note that since $\Pi_\infty^{\mathrm{cond}}(S) \simeq \mathrm{Gal}(S)$ is 1-truncated, the second claim follows from the first by taking homotopy condensed sets. ☐

5.7 Corollary. *Let k be a field with separable closure $\bar{k}$, let X be a qcqs k-scheme, and fix a geometric point $\bar{x} \to X_{\bar{k}}$. If $\pi_0^{\mathrm{cond}}(X_{\bar{k}}) = 1$, then the sequence of condensed groups*

$$1 \longrightarrow \pi_1^{\mathrm{cond}}(X_{\bar{k}}, \bar{x}) \longrightarrow \pi_1^{\mathrm{cond}}(X, \bar{x}) \longrightarrow \mathrm{Gal}_k \longrightarrow 1$$

is exact.

5.8 Remark. By Corollary 4.19, the hypotheses of Corollary 5.7 are satisfied if X is geometrically connected and $X_{\bar{k}}$ has finitely many irreducible components.

As an application of the fundamental fiber sequence and Corollary 4.33, we compute of the condensed homotopy type of rings of continuous functions to $\mathbf{R}$:

5.9 Corollary. *Let T be a compact Hausdorff space. Then there is a natural equivalence of condensed anima*

$$\Pi_\infty^{\mathrm{cond}}(\mathrm{Spec}(\mathrm{C}(T, \mathbf{R}))) \simeq T \times \mathrm{BGal}_{\mathbf{R}} \ .$$

Proof. As explained in Lemma A.26, the natural ring homomorphism $\mathrm{C}(T, \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C} \to \mathrm{C}(T, \mathbf{C})$ is an isomorphism. Hence by the fundamental fiber sequence

$$\Pi_\infty^{\mathrm{cond}}(\mathrm{Spec}(\mathrm{C}(T, \mathbf{C}))) \to \Pi_\infty^{\mathrm{cond}}(\mathrm{Spec}(\mathrm{C}(T, \mathbf{R}))) \to \mathrm{BGal}_{\mathbf{R}}$$

of Corollary 5.6, we just have to show that action of $\mathrm{Gal}_{\mathbf{R}}$ on $\Pi_\infty^{\mathrm{cond}}(\mathrm{Spec}(\mathrm{C}(T, \mathbf{C})))$ is trivial. By Theorem 4.28, we have natural identifications

$$\Pi_\infty^{\mathrm{cond}}(\mathrm{Spec}(\mathrm{C}(T, \mathbf{C}))) \simeq \mathrm{MSpec}(\mathrm{C}(T, \mathbf{C})) \simeq T \ .$$

Thus it suffices to show that map on maximal spectra

$$\mathrm{MSpec}(\mathrm{C}(T, \mathbf{C})) \to \mathrm{MSpec}(\mathrm{C}(T, \mathbf{C}))$$

induced by complex conjugation is the identity. To see this, note that by Theorem A.32, each maximal ideal is given by all functions $T \to \mathbf{C}$ that vanish at some fixed $t \in T$, and a function vanishes at a point if and only if its conjugate does. ☐

5.2 Geometric and homotopy-theoretic fibers

Let $f : X \to S$ be a smooth and proper morphism of schemes. The goal of this subsection is is to show that, up to suitable completion, the homotopy-theoretic fiber of the induced map $\Pi_\infty^{\mathrm{cond}}(f) : \Pi_\infty^{\mathrm{cond}}(X) \to \Pi_\infty^{\mathrm{cond}}(S)$ agrees with the condensed homotopy type of the scheme-theoretic fiber.

5.10 Notation. For a morphism of schemes $f : X \to S$ and a geometric point $\bar{s} \to S$, we denote by

$$X_{(\bar{s})} \coloneqq X \times_S S_{(\bar{s})}$$

the *Milnor ball of f at $\bar{s}$* . Here $S_{(\bar{s})}$ denotes the strict localization at $\bar{s}$.

5.11 Recollection (Σ-completion)**.** Let Σ be a nonempty set of prime numbers.

(1) We write $\mathbf{Ani}_\Sigma \subset \mathbf{Ani}_\pi$ for the full subcategory spanned by those π-finite anima all of whose homotopy groups are Σ-groups (i.e., their order is a product of elements of Σ).

(2) The inclusion $\mathrm{Pro}(\mathbf{Ani}_\Sigma) \hookrightarrow \mathrm{Pro}(\mathbf{Ani}_\pi)$ admits a left adjoint $(-)^\wedge_\Sigma$ that we refer to as *Σ-completion*.

(3) We also write $(-)^\wedge_\Sigma : \mathrm{Cond}(\mathbf{Ani}) \to \mathrm{Pro}(\mathbf{Ani}_\Sigma)$ for the left adjoint of the inclusion

$$\mathrm{Pro}(\mathbf{Ani}_\Sigma) \hookrightarrow \mathrm{Pro}(\mathbf{Ani}_\pi) \hookrightarrow \mathrm{Cond}(\mathbf{Ani}) .$$

As a consequence of the exodromy description of the condensed homotopy type, we can apply a profinite version of Quillen's Theorem B, see §C.2, to get the following:

5.12 Theorem. *Let $f : X \to S$ be a smooth and proper morphism between qcqs schemes and let $\bar{s} \to S$ be a geometric point. Let Σ be a set of primes invertible on S. Then the induced map*

$$\Pi_\infty^{\mathrm{cond}}(X_{\bar{s}}) \to \mathrm{fib}_{\bar{s}}(\Pi_\infty^{\mathrm{cond}}(f))$$

becomes an equivalence after Σ-completion.

Proof. We want to apply Theorem C.7 to the functor $\mathrm{Gal}(f) : \mathrm{Gal}(X) \to \mathrm{Gal}(S)$ induced by f. To verify that the assumptions of Theorem C.7 are satisfied, we need to see that for any specialization $\eta : \bar{t}' \to \bar{t}$ in S, the induced map

$$\mathrm{B}^{\mathrm{cond}}(\mathrm{Gal}(X)_{\bar{t}/}) \to \mathrm{B}^{\mathrm{cond}}(\mathrm{Gal}(X)_{\bar{t}'/}) \tag{5.13}$$

becomes an equivalence after Σ-completion.

Recall that by [8, Corollary 12.4.5], we have a natural equivalence of underlying ∞-categories

$$\mathrm{Gal}(S_{(\bar{t})}) \xrightarrow{\sim} \mathrm{Gal}(S)_{\bar{t}/} . \tag{5.14}$$

Using Observation 6.5 below, one can show that this equivalence refines to an equivalence of condensed ∞-categories, see [81, Proposition 7.3.3.7] for more details. Furthermore, [36, Proposition 2.4] implies, that the natural functor

$$\mathrm{Gal}(X_{(\bar{t})}) \to \mathrm{Gal}(X)_{\bar{t}/} ,$$

induced by the equivalence (5.14), is an equivalence of condensed ∞-categories as well. Thus by Lemma 3.14, the Σ-completion of the map (5.13) identifies with the specialization map

$$\widehat{\Pi}_{\infty}^{\text{ét}}(X_{(\bar{t})})_{\Sigma}^{\wedge} \to \widehat{\Pi}_{\infty}^{\text{ét}}(X_{(\bar{t}')})_{\Sigma}^{\wedge} .$$

By [35, Proposition 2.49], this specialization map is an equivalence. Thus, Theorem C.7 implies that the natural map $\Pi_{\infty}^{\text{cond}}(X_{(\bar{s})}) \to \text{fib}_{\bar{s}}(\Pi_{\infty}^{\text{cond}}(f))$ becomes an equivalence after Σ-completion. Finally, note that by Lemma 3.14 and [35, Corollary 2.39], the natural map

$$\Pi_{\infty}^{\text{cond}}(X_{\bar{s}}) \to \Pi_{\infty}^{\text{cond}}(X_{(\bar{s})})$$

becomes an equivalence after Σ-completion. □

5.15 Remark. In the setting of Theorem 5.12, the canonical map $\Pi_{\infty}^{\text{cond}}(X_{\bar{s}}) \to \text{fib}_{\bar{s}}(\Pi_{\infty}^{\text{cond}}(f))$ is not generally an equivalence before Σ-completion. The reason why this fails is that the proper and smooth base change theorems do not hold for arbitrary proétale sheaves; they only hold for constructible étale sheaves.

5.16 Remark. Theorem 5.12 is an analogue of Friedlander's result [22, Theorem 3.7]. Since we do not have to require that the base S be normal, at the cost of working with a more complicated homotopy type, our result holds in a more general setup. However, since the Σ-completion functor does not preserve fiber sequences, it is also not immediate how to recover Friedlander's result from ours.

6 Integral Descent

The goal of this section is to prove that the condensed homotopy type satisfies integral hyperdescent. Let us start by formulating what we mean by this more precisely.

6.1 Definition. Let X be a scheme and $\mathcal{C}$ an ∞-category.

(1) We call an augmented simplical object $X_{\bullet} \to X$ an *integral hypercover* if for each $n \geq 0$, the morphism $X_n \to X$ is integral and $X_0 \to X$ and $X_n \to (\text{cosk}_{n-1}(X_{\bullet}))_n$ are surjective.

(2) We call a functor $F : \mathbf{Sch}^{\text{qcqs}} \to \mathcal{C}$ a *hypercomplete integral cosheaf* if F sends integral hypercovers to colimit diagrams.

The main goal of §6.1 is to show that $\Pi_{\infty}^{\text{cond}}(-)$ is a hypercomplete integral cosheaf, which we achieve in Corollary 6.16. In fact, our methods will show that already $\text{Gal}(-)$ is a hypercomplete integral cosheaf of condensed categories. In §6.2, we use some of the results in this section to characterize those morphisms of schemes, for which the étale ∞-topos is compatible with base change; this included integral morphisms.

6.1 Integral morphisms and right fibrations

In this subsection, we show that for an integral morphism of schemes, the induced functor on Galois categories is a right fibration of condensed categories. We begin by recalling the notion of a right fibration of condensed ∞-categories:

6.2 Definition. We say that a functor of condensed ∞-categories $f : \mathcal{C} \to \mathcal{D}$ is a *right fibration* if and only if the commutative square

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{cond}}([1], \mathcal{C}) \arrow[r, "f \circ -"] \arrow[d, "\mathrm{ev}_1"'] & \mathrm{Fun}^{\mathrm{cond}}([1], \mathcal{D}) \arrow[d, "\mathrm{ev}_1"] \\
\mathcal{C} \arrow[r, "f"'] & \mathcal{D}
\end{tikzcd}$$

is a cartesian square in $\mathrm{Cond}(\mathbf{Cat}_\infty)$.

6.3 Remark. Definition 6.2 is a special case of the notion of a right fibration of simplicial objects in a general ∞-topos $\mathcal{B}$, as introduced in [57, Definition 4.1.1]. In particular it follows from the discussion in *loc. cit.* that right fibrations in $\mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathrm{Cond}(\mathbf{Ani}))$ are the right class in an orthogonal factorization system. The left class consists of the *final* maps, i.e., the smallest saturated class which contains all maps of the form $\{n\} \times S \hookrightarrow [n] \times S$ for $n \in \mathbf{N}$ and $S \in \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$. See [57, Lemma 4.1.2].

6.4 Remark. A functor $f : \mathcal{C} \to \mathcal{D}$ of condensed ∞-categories is a right fibration if and only if for every profinite set S, the functor $f(S) : \mathcal{C}(S) \to \mathcal{D}(S)$ is a right fibration of ∞-categories. Indeed, the square in Definition 6.2 is cartesian if and only if this is true after evaluation at every profinite set S. Under the equivalence $\mathrm{Fun}^{\mathrm{cond}}([1], \mathcal{C})(S) \simeq \mathrm{Fun}([1], \mathcal{C}(S))$, the claim then follows by the characterization of right fibrations via a corresponding cartesian square, see [14, Proposition 3.4.5].

In the cases we care about, being a right fibration can often be detected on the level of underlying categories, which we deduce from the following observation.

6.5 Observation. Recall from [SAG, Theorem E.3.1.6] that the functor

$$\lim : \mathrm{Pro}(\mathbf{Ani}_\pi) \to \mathbf{Ani}$$

is conservative. It follows that the functor $\lim_* : \mathrm{Cat}(\mathrm{Pro}(\mathbf{Ani}_\pi)) \to \mathbf{Cat}_\infty$ given by postcomposition with lim is also conservative.

6.6 Lemma. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor in* $\mathrm{Cat}(\mathrm{Pro}(\mathbf{Ani}_\pi))$ *considered as a functor of condensed ∞-categories. If the underlying functor of ∞-categories is a right fibration, then f is a right fibration of condensed ∞-categories.*

Proof. By definition, f is a right fibration if and only if the induced map

$$\mathrm{Fun}^{\mathrm{cond}}([1], \mathcal{C}) \to \mathrm{Fun}^{\mathrm{cond}}([1], \mathcal{D}) \times_{\mathcal{D}} \mathcal{C} \tag{6.7}$$

is an equivalence of condensed ∞-categories. Since $\mathcal{C}$ and $\mathcal{D}$ are in $\mathrm{Cat}(\mathrm{Pro}(\mathbf{Ani}_\pi))$, it follows that $\mathrm{Fun}^{\mathrm{cond}}([1], \mathcal{C})$ and $\mathrm{Fun}^{\mathrm{cond}}([1], \mathcal{D})$ are also in $\mathrm{Cat}(\mathrm{Pro}(\mathbf{Ani}_\pi))$. Thus, by Observation 6.5, the comparison map (6.7) is an equivalence if and only if it an equivalence on underlying ∞-categories. Since taking underlying ∞-categories commutes with pullbacks, this proves the claim. ☐

By Recollection 3.31, we immediately deduce the following.

6.8 Corollary. *Let $f : X \to Y$ be a morphism of qcqs schemes. Then the induced functor*

$$\mathrm{Gal}(f) : \mathrm{Gal}(X) \to \mathrm{Gal}(Y)$$

is a right fibration of condensed categories if and only if this is true on the underlying categories.

6.9 Proposition. *Let $f : X \to Y$ be an integral morphism of qcqs schemes. Then the induced functor*

$$\mathrm{Gal}(f) \colon \mathrm{Gal}(X) \to \mathrm{Gal}(Y)$$

is a right fibration of condensed categories.

Proof. By Corollary 6.8, it suffices to check this on underlying categories. The statement about underlying categories appears in [8, Proposition 14.1.6]; for the convenience of the reader, we give a quick proof here.

Throughout the proof, we simply write $\mathrm{Gal}(-)$ for the underlying category as well. By [STK, Tag 09YZ], any integral morphism $f : X \to Y$ with Y qcqs can be written as $f = \lim_i f_i$ for some cofiltered system of *finite* morphisms $f_i : X_i \to Y$. Since right fibrations are stable under limits, by the continuity of étale ∞-topoi [SGA 4_{II}, Éxpose VII, Lemma 5.6; 16, Proposition 3.10], we may assume that f is finite. Since $\mathrm{Gal}(X)$ and $\mathrm{Gal}(Y)$ are 1-categories, by [Ker, Tag 015H] it suffices to show that any lifting problem of the form

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[d, hook] & \mathrm{Gal}(X) \arrow[d, "\mathrm{Gal}(f)"] \\
{[1]} \arrow[r, "s"'] \arrow[ur, dotted, "\exists!?"] & \mathrm{Gal}(Y).
\end{tikzcd}$$

has a *unique* solution. Writing $\bar{y}$ for the source of the map s, this diagram factors as

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[d, hook] & \mathrm{Gal}(Y)_{\bar{y}/} \times_{\mathrm{Gal}(Y)} \mathrm{Gal}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Gal}(X) \arrow[d, "\mathrm{Gal}(f)"] \\
{[1]} \arrow[r] \arrow[ur, dotted, "\exists!?"] \arrow[rr, bend right, "s"'] & \mathrm{Gal}(Y)_{\bar{y}/} \arrow[r] & \mathrm{Gal}(Y)\,,
\end{tikzcd}$$

and it suffices to show that this induced lifting problem has a unique solution.

By [8, Corollary 12.4.5] and [36, Corollary 2.4], we can identify

$$\mathrm{Gal}(Y)_{\bar{y}/} \simeq \mathrm{Gal}(Y_{(\bar{y})}) \qquad \text{and} \qquad \mathrm{Gal}(X) \times_{\mathrm{Gal}(Y)} \mathrm{Gal}(Y_{(\bar{y})}) \simeq \mathrm{Gal}(X \times_Y Y_{(\bar{y})})\,.$$

Moreover, since $f : X \to Y$ is finite, by [STK, Tag 04GH] we have a coproduct decomposition $X \times_Y Y_{(\bar{y})} = \coprod_{\bar{x}_i \in f^{-1}(\bar{y})} X_{(\bar{x}_i)}$. Now the map

$$\{1\} \to \mathrm{Gal}(Y_{(\bar{y})}) \times_{\mathrm{Gal}(Y)} \mathrm{Gal}(X) \simeq \coprod_i \mathrm{Gal}(X_{(\bar{x}_i)})$$

factors through $\mathrm{Gal}(X_{(\bar{x}_{i_0})})$ for some i_0. Hence, writing $\bar{x} \coloneqq \bar{x}_{i_0}$, we finally arrive at a lifting problem of the form

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[d, hook] & \mathrm{Gal}(X_{(\bar{x})}) \arrow[r] \arrow[d] & \mathrm{Gal}(X) \arrow[d, "\mathrm{Gal}(f)"] \\
{[1]} \arrow[r] \arrow[ur, dotted, "\exists!?"] \arrow[rr, bend right, "s"'] & \mathrm{Gal}(Y_{(\bar{y})}) \arrow[r] & \mathrm{Gal}(Y).
\end{tikzcd}$$

Here, existence and uniqueness of a lift is clear. Let $\bar{y}'$ be the target of the map s, determined by $\{1\} \to \mathrm{Gal}(X_{(\bar{x})})$. Note that $\bar{x}$ is the initial object of $\mathrm{Gal}(X_{(\bar{x})}) \simeq \mathrm{Gal}(X)_{\bar{x}/}$, and also the only object lifting $\bar{y}$. So if there exists a lift, it has to be the unique map from $\bar{x} \to \bar{x}'$ for $\bar{x}'$ the lift of $\bar{y}'$. Since $\bar{y}$ is the initial object of $\mathrm{Gal}(Y_{(\bar{y})}) \simeq \mathrm{Gal}(Y)_{\bar{y}/}$, it is clear that $\bar{x} \to \bar{x}'$ actually lifts the map $s\colon \bar{y} \to \bar{y}'$ we started with. □

6.10 Corollary (Künneth formula for integral morphisms)**.** *Let $X \to Y$ be an integral morphism of qcqs schemes. Then for any qcqs scheme Y' and morphism $Y' \to Y$ the natural functor*

$$\mathrm{Gal}(X \times_Y Y') \to \mathrm{Gal}(X) \times_{\mathrm{Gal}(Y)} \mathrm{Gal}(Y')$$

is an equivalence.

Proof. As integral morphisms and right fibrations are stable under pullbacks, by Proposition 6.9 both functors

$$\mathrm{Gal}(\mathrm{pr}_1)\colon \mathrm{Gal}(X \times_Y Y') \to \mathrm{Gal}(Y') \qquad \text{and} \qquad \mathrm{pr}_1\colon \mathrm{Gal}(X) \times_{\mathrm{Gal}(Y)} \mathrm{Gal}(Y') \to \mathrm{Gal}(Y')$$

are right fibrations. Therefore, by [Ker, Tag 01VE] it suffices to see that the natural functor

$$\mathrm{Gal}(X \times_Y Y') \to \mathrm{Gal}(X) \times_{\mathrm{Gal}(Y)} \mathrm{Gal}(Y')$$

becomes an equivalence after taking fibers over any $\bar{y}' \in \mathrm{Gal}(Y')$. This holds by [35, Corollary 2.4]. □

6.11 Lemma. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a morphism in* $\mathrm{Cat}(\mathrm{Pro}(\mathbf{Ani}_\pi))$. *Then f is surjective as a functor of condensed ∞-categories (i.e., for all $S \in$ **Extr**, the functor $\mathcal{C}(S) \to \mathcal{D}(S)$ is surjective) if and only if the induced functor on underlying ∞-categories $f(*)\colon \mathcal{C}(*) \to \mathcal{D}(*)$ is surjective.*

6.12 Observation. The inclusion $\mathrm{Cond}(\mathbf{Ani}) \to \mathrm{Cond}(\mathbf{Cat}_\infty)$ also admits a right adjoint. We denote this right adjoint by $(-)^{\simeq}$.

Proof of Lemma 6.11. First, by definition, if f is a surjective functor of condensed ∞-categories, then $f(*)\colon \mathcal{C}(*) \to \mathcal{D}(*)$ is surjective. Conversely, if $f(*)\colon \mathcal{C}(*) \to \mathcal{D}(*)$ is surjective, then it follows from [SAG, Corollary E.4.6.3] that the induced map $\mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is an effective epimorphism in $\mathrm{Pro}(\mathbf{Ani}_\pi) \subset \mathrm{Cond}(\mathbf{Ani})$. Now let $S \in$ **Extr**. Since any map $S \to \mathcal{D}$ in $\mathrm{Cond}(\mathbf{Cat}_\infty)$ factors through $\mathcal{D}^{\simeq}$ and S is projective in $\mathrm{Cond}(\mathbf{Ani})$ it follows that we can find a lift in the diagram

$$\begin{tikzcd} & \mathcal{C}^{\simeq} \arrow[d, "f"] \\ S \arrow[r] \arrow[ur, dotted] & \mathcal{D}^{\simeq} \end{tikzcd}$$

which completes the proof. □

6.13 Corollary. *Let $f\colon X \to Y$ be a surjective morphism of qcqs schemes. Then the functor of condensed categories $\mathrm{Gal}(f)\colon \mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ is surjective.*

Proof of Corollary 6.13. By Lemma 6.11, we just need to see that the induced functor on categories of points $\mathrm{Gal}(X)(*) \to \mathrm{Gal}(Y)(*)$ is surjective. Since any point of $X_{\text{ét}}$ is represented by a geometric point $\bar{x} \to X$, it is clear. □

Right fibrations automatically satisfy descent in the following sense:

6.14 Definition. An augmented simplicial ∞-category $\mathcal{C}_\bullet \to \mathcal{C}$ is a *hypercover* if for each $n \in \mathbf{N}$, the induced functor $\mathcal{C}_n \to (\mathrm{cosk}_{n-1}(\mathcal{C}_\bullet))_n$ is surjective.

6.15 Lemma. *Let $\mathcal{C}_\bullet \to \mathcal{C}$ be a hypercover in $\mathbf{Cat}_\infty$, and assume that for each $n \in \mathbf{N}$, the induced functor $\mathcal{C}_n \to \mathcal{C}$ is a right fibration. Then $\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{C}_\bullet \xrightarrow{\sim} \mathcal{C}$.*

Proof. By straightening-unstraightening, our given hypercover translates to a hypercover of the terminal object in the ∞-category $\mathrm{RFib}(\mathcal{C}) \simeq \mathrm{PSh}(\mathcal{C})$ of right fibrations over $\mathcal{C}$. Furthermore, the inclusion $\mathrm{RFib}(\mathcal{C}) \subset \mathbf{Cat}_{\infty,/\mathcal{C}}$ preserves limits and colimits (the case of limits is clear as right fibrations are defined via a lifting property, for colimits see [66, Corollary A.5]). Since $\mathrm{RFib}(\mathcal{C})$ is a presheaf ∞-topos and therefore hypercomplete, the claim follows. ☐

We can now deduce the desired descent results.

6.16 Corollary.

(1) *The functor* $\mathrm{Gal} : \mathbf{Sch}^{\mathrm{qcqs}} \to \mathrm{Cond}(\mathbf{Cat}_\infty)$ *is a hypercomplete integral cosheaf.*

(2) *The functor* $(-)^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}} : (\mathbf{Sch}^{\mathrm{qcqs}})^{\mathrm{op}} \to \mathbf{Cat}_\infty$ *with functoriality given by pullbacks is an integral hypersheaf.*

(3) *The functor* $\Pi^{\mathrm{cond}}_\infty : \mathbf{Sch}^{\mathrm{qcqs}} \to \mathrm{Cond}(\mathbf{Ani})$ *is a hypercomplete integral cosheaf.*

Proof. By [80, Theorem 1.2], we have a natural equivalence

$$X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}} \simeq \mathrm{Fun}^{\mathrm{cts}}(\mathrm{Gal}(X), \mathbf{Cond}(\mathbf{Ani})) ,$$

hence second assertion is an immediate consequence of the first. By Proposition 3.36, the third assertion is also an immediate consequence of the first. Thus, we only need to prove the first assertion.

Using Corollary 6.10, it follows that for any integral hypercover $X_\bullet \to X$ and $n \in \mathbf{N}$, the canonical map

$$\mathrm{Gal}(\mathrm{cosk}_{n-1}(X_\bullet)_n) \to \mathrm{cosk}_{n-1}(\mathrm{Gal}(X_\bullet))_n$$

is an equivalence. Thus, Proposition 6.9 and Corollary 6.13 imply that $\mathrm{Gal}(X_\bullet)$ is a hypercover of right fibrations of condensed categories. Since sifted colimits are computed pointwise in the ∞-category $\mathrm{Cond}(\mathbf{Cat}_\infty) = \mathrm{Fun}^\times(\mathbf{Extr}^{\mathrm{op}}, \mathbf{Cat}_\infty)$, the claim follows by combining Remark 6.4 and Lemma 6.15. ☐

We can also recover the schematic description of the over category $\mathrm{Gal}(X)_{/\bar{x}}$ given in [8, Corollary 12.4.5]:[5]

6.17 Corollary. *Let X be a qcqs scheme, let $\bar{x} \to X$ be a geometric point, and let $X^{(\bar{x})}$ denote the* strict normalization *of X at $\bar{x}$ in the sense of [8, Notation 12.4.2]. Then the natural integral morphism $f : X^{(\bar{x})} \to X$ induces an equivalence of condensed categories*

$$\mathrm{Gal}(X^{(\bar{x})}) \xrightarrow{\sim} \mathrm{Gal}(X)_{/\bar{x}} .$$

---

[5] The description of the under categories of $\mathrm{Gal}(X)$ in terms of strict henselizations in *loc. cit.* is immediate from the definition. The description of over categories in terms of strict normalizations is less obvious, so we decided to include an argument here.

Proof. Since the morphsism f is integral, by Proposition 6.9 the functor of condensed categories $\mathrm{Gal}(f)$ is a right fibration. Hence for $\bar{x} \colon * \to \mathrm{Gal}(X^{(\bar{x})}) \to \mathrm{Gal}(X)$, the induced functor

$$f_{/\bar{x}} \colon \mathrm{Gal}(X^{(\bar{x})})_{/\bar{x}} \to \mathrm{Gal}(X)_{/\bar{x}}$$

is an equivalence of condensed categories. The condensed category $\mathrm{Gal}(X^{(\bar{x})})$ already has a terminal object induced by the generic point of $X^{(\bar{x})}$, which is given by $\bar{x} \to X^{(\bar{x})}$, cf. [56, Theorem 2.4.21]. We conclude using that

$$\mathrm{Gal}(X^{(\bar{x})}) \simeq \mathrm{Gal}(X^{(\bar{x})})_{/\bar{x}} \simeq \mathrm{Gal}(X)_{/\bar{x}} . \qquad \square$$

Finally, using some of the machinery developed in [57], we can also deduce integral basechange for proétale hypersheaves. We do not need this in the rest of this article, but it might be of independent interest.

6.18 Proposition. *Let*

$$\begin{tikzcd} X' \arrow[r, "q"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "p"'] & Y \end{tikzcd}$$

be a cartesian square of qcqs schemes where f is integral. Then the induced square

$$\begin{tikzcd} (X')^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}} \arrow[r, "q_*"] \arrow[d, "g_*"'] & X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}} \arrow[d, "f_*"] \\ (Y')^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}} \arrow[r, "p_*"'] & Y^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}} \end{tikzcd}$$

is horizontally left adjointable, i.e., the natural exchange transformation $p^* f_* \to g_* q^*$ is an equivalence.

Proof. By [80, Corollary 1.2], this square is identified with the square

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{cts}}(\mathrm{Gal}(X'), \mathbf{Cond}(\mathbf{Ani})) \arrow[r, "\mathrm{Gal}(q)_*"] \arrow[d, "\mathrm{Gal}(g)_*"'] & \mathrm{Fun}^{\mathrm{cts}}(\mathrm{Gal}(X), \mathbf{Cond}(\mathbf{Ani})) \arrow[d, "\mathrm{Gal}(f)_*"] \\ \mathrm{Fun}^{\mathrm{cts}}(\mathrm{Gal}(Y'), \mathbf{Cond}(\mathbf{Ani})) \arrow[r, "\mathrm{Gal}(p)_*"'] & \mathrm{Fun}^{\mathrm{cts}}(\mathrm{Gal}(Y), \mathbf{Cond}(\mathbf{Ani})) . \end{tikzcd}$$

Since f is integral, Proposition 6.9 shows that $\mathrm{Gal}(f)$ is a right fibration, and Corollary 6.10 shows that the natural map $\mathrm{Gal}(X') \to \mathrm{Gal}(X) \times_{\mathrm{Gal}(Y)} \mathrm{Gal}(Y')$ is an equivalence. Because right fibrations of condensed ∞-categories are *proper* functors [57, Proposition 4.4.7], the the above square is horizontally left adjointable. $\square$

6.2 Digression: strongly künnethable morphisms of schemes

We conclude this section by explaining at what level of generality the Künneth formula for étale ∞-topoi (equivalently, Corollary 6.10) holds.

6.19 Definition. We call a morphism of schemes $X \to Y$ *strongly künnethable* if for any morphism $Y' \to Y$ the induced map

$$(X \times_Y Y')_{\text{ét}} \to X_{\text{ét}} \times_{Y_{\text{ét}}} Y'_{\text{ét}}$$

is an equivalence.

6.20 Remark. Since all ∞-topoi involved in Definition 6.19 are 1-localic, being strongly künnethable is equivalent to the canonical geometric morphism

$$(X \times_Y Y')_{\text{ét},\leq 0} \to X_{\text{ét},\leq 0} \times_{Y_{\text{ét},\leq 0}} Y'_{\text{ét},\leq 0}$$

of 1-topoi being an equivalence.

6.21 Proposition. *Let $f : X \to Y$ be a morphism of finite presentation. Then f is strongly künnethable if and only if it is quasi-finite.*

Proof. Let us first assume that f is quasi-finite. Since open immersions are strongly künnethable by [HTT, Remark 6.3.5.8], we may immediately reduce to the case where X, Y, and Y' are affine. Applying Zariski's main theorem, we can factor f as an open immersion followed by a finite morphism. Thus we may assume that f is finite.

We have to check that the induced map

$$(X \times_Y Y')_{\text{ét},\leq 0} \to X_{\text{ét},\leq 0} \times_{Y_{\text{ét},\leq 0}} Y'_{\text{ét},\leq 0} \tag{6.22}$$

is an equivalence. By Corollary 6.10, it induces an equivalence of categories of points. Furthermore it follows from the site-theoretic description of the fiber product of topoi [45, Exposé XI, §3] that (6.22) is a coherent geometric morphism of coherent topoi. Thus, the Makkai–Reyes conceptual completeness theorem [SAG, Theorem A.9.0.6] implies that this geometric morphism is an equivalence.

For the converse, assume that f is not quasi-finite. Then at least one geometric fiber of f is not quasi-finite. Since taking geometric fibers is compatible with taking étale ∞-topoi [36, Proposition 2.3], we may reduce to the case where $Y = \operatorname{Spec}(k)$ is the spectrum of a separably closed field k. Furthermore, we may always modify X by quasi-finite maps to reduce to the case where X is integral of dimension at least 1. By Noether normalization, there exists a finite surjective map $h : X \to \mathbf{A}^n_k$. Let $X_\bullet \to \mathbf{A}^n_k$ denote the Čech nerve of h. Now if f were strongly künnethable, then since the maps $X_m \to \operatorname{Spec}(k)$ are the composite of a finite map $d_0 : X_m \to X$ and f, it would follow that also all maps $X_m \to \operatorname{Spec}(k)$ would be strongly künnethable as well. Thus for every k-scheme Y' and every $m \geq 0$, the induced map

$$\operatorname{Gal}(X_m \times Y') \to \operatorname{Gal}(X_m) \times \operatorname{Gal}(Y')$$

would be an equivalence. But by integral descent (Corollary 6.16), after passing to the colimit over $\mathbf{\Delta}^{\text{op}}$, this would imply that the canonical map

$$\operatorname{Gal}(\mathbf{A}^n_k \times Y') \to \operatorname{Gal}(\mathbf{A}^n_k) \times \operatorname{Gal}(Y')$$

is an equivalence.

Thus we may assume that $X = \mathbf{A}^n_k$ and therefore even that $X = \mathbf{A}^1_k$. Now let $Z = \mathbf{A}^1_k$ as well. This would imply that the canonical map

$$\operatorname{Gal}(\mathbf{A}^2_k) \to \operatorname{Gal}(\mathbf{A}^1_k) \times \operatorname{Gal}(\mathbf{A}^1_k)$$

is an equivalence. In particular, it would induce an equivalence on underlying posets and thus an isomorphism of specialization posets

$$(\mathbf{A}_k^2)_{\mathrm{zar}}^{\leq} \to (\mathbf{A}_k^1)_{\mathrm{zar}}^{\leq} \times (\mathbf{A}_k^1)_{\mathrm{zar}}^{\leq} ,$$

which is a contradiction. □

Part II

The condensed fundamental group

The purpose of this part is to analyze the fundamental group of the condensed homotopy type and its relationship to the étale and proétale fundamental groups. We start by showing that, surprisingly, $\pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}})$ is nontrivial (see Corollary 7.4). This can be viewed as saying that there exists a nontrivial proétale local system of *condensed* rings on $\mathbf{A}^1_{\mathbf{C}}$.

In § 7, we show that a mild quotient of the condensed fundamental group of $\mathbf{A}^1_{\mathbf{C}}$ indeed becomes trivial. Specifically, Clausen and Scholze introduced a localization $A \mapsto A^{\mathrm{qs}}$ of the category of condensed sets called the *quasiseparated quotient* [69, Lecture VI]. For topological groups, this is analogous to the Hausdorff quotient. We show that if X is a topologically noetherian scheme that is geometrically unibranch, then there is a natural isomorphism of condensed groups

$$\pi_1^{\mathrm{cond}}(X, \bar{x})^{\mathrm{qs}} \xrightarrow{\sim} \pi_1^{\mathrm{\acute{e}t}}(X, \bar{x}),$$

see Theorem 7.17. Under mild hypotheses on the scheme (e.g., being Nagata), we also prove a van Kampen formula for the quasiseparated quotient of the condensed fundamental group that only involves topological free products, topological quotients, and the étale fundamental group of the normalization, see Theorem 7.35.

In §8, we turn to the relationship between the condensed fundamental group and the proétale fundamental group introduced by Bhatt and Scholze [10, §7]. One of the special features of $\pi_1^{\mathrm{pro\acute{e}t}}(X)$ is that it is a *Noohi group*. We show that if X is topologically noetherian, the *Noohi completion* (suitably extended to condensed groups) of $\pi_1^{\mathrm{cond}}(X)$ recovers $\pi_1^{\mathrm{pro\acute{e}t}}(X)$, see Theorem 8.12.

7 The quasiseparated quotient of the condensed fundamental group

In § 7.1, we begin by using the Galois category description of the condensed homotopy type to show that $\pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}})$ is nontrivial. The rest of the section is dedicated to studying the quasiseparated quotient of $\pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}})$. In § 7.2, we recall the basics on quasiseparated quotients of condensed sets and prove some fundamental results about the quasiseparated quotient. In § 7.3, we show that the quasiseparated quotient of π_1^{cond} of a geometrically unibranch and topologically noetherian scheme recovers $\pi_1^{\mathrm{\acute{e}t}}$. In §7.4, we prove a van Kampen formula for the quasiseparated quotient of the condensed fundamental group, see Theorem 7.35.

7.1 $\pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}})$ is nontrivial

In this subsection, we show that π_1^{cond} can behave wildly, even in geometrically very simple situations. For simplicity, we work over the complex numbers $\mathbf{C}$ in this section. However, we believe that many analogous statements hold over any algebraically closed field of characteristic 0.

7.1 Notation. For a topological group G and an (abstract) subgroup $H < G$, let H^{nc} denote the group-theoretic *normal closure* of H in G. Let

$$H^{\mathrm{tnc}} := \overline{H^{\mathrm{nc}}}$$

be the *topological normal closure* of H in G, i.e., the smallest *closed* normal subgroup of G containing H or, equivalently, the topological closure of H^{nc} in G.

7.2 Proposition. *Let $S \subset \mathbf{C}$ be a subset. Let us write*

$$\mathbf{A}^1_{\mathbf{C}} \smallsetminus S \coloneqq \operatorname{Spec}(\mathbf{C}[t][(t-a)^{-1} \mid a \in S]) .$$

Let $\widehat{\mathrm{Fr}}_{\mathbf{C}}$ be the free profinite group on the underlying set of $\mathbf{C}$. Let N_S be the abstract normal subgroup of $\widehat{\mathrm{Fr}}_{\mathbf{C}}$ generated by $\widehat{\mathbf{Z}}(a)$ for all $a \in \mathbf{C} \smallsetminus S$. Write η for the generic point of $\mathbf{A}^1_{\mathbf{C}}$ and $\bar{\eta}$ for the geometric generic point induced by choosing an algebraic closure of $\mathbf{C}(T)$. There is a short exact sequence of (abstract) groups

$$1 \to N_S \to \widehat{\mathrm{Fr}}_{\mathbf{C}} \to \pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}} \smallsetminus S, \bar{\eta})(*) \to 1 .$$

To prove Proposition 7.2, we make use of an alternative description of $\mathrm{BGal}(X)(*)$ provided by the following lemma.

7.3 Lemma. *Let X be a qcqs scheme. Then there is a natural equivalence*

$$\mathrm{BGal}(X)(*) \simeq \operatorname*{colim}_{\mathrm{sd}(X^{\leq}_{\mathrm{zar}})^{\mathrm{op}}} \mathrm{Dec}(\mathrm{Gal}(X)) .$$

Proof. Let us simplify notation and write $\mathrm{Gal}(X)$ instead of $\mathrm{Gal}(X)(*)$. Then the functor $\mathrm{Gal}(X) \to X^{\leq}_{\mathrm{zar}}$ is a stratified space and under the equivalence [8, Theorem 2.7.4] it corresponds to a functor

$$\mathrm{Dec}(\mathrm{Gal}(X)) \colon \ \mathrm{sd}(X^{\leq}_{\mathrm{zar}})^{\mathrm{op}} \to \mathbf{Ani} .$$

Here $\mathrm{sd}(X^{\leq}_{\mathrm{zar}})$ denotes the subdivision poset of $X^{\leq}_{\mathrm{zar}}$, see [8, Notation 1.1.8]. Furthermore, observe that $\operatorname{colim} \mathrm{Dec}(\mathrm{Gal}(X)) \simeq \mathrm{BGal}(X)$. Indeed, the composite functor

$$\begin{tikzcd}
\mathbf{Ani} \arrow[r, "- \times X^{\leq}_{\mathrm{zar}}"] & \mathbf{Str}_{X^{\leq}_{\mathrm{zar}}} \arrow[r, "\sim"] & \mathrm{Fun}(\mathrm{sd}(X^{\leq}_{\mathrm{zar}})^{\mathrm{op}}, \mathbf{Ani})
\end{tikzcd}$$

preserves colimits and thus coincides with the constant diagram functor from **Ani** whose left adjoint is given by taking the colimit. ☐

Proof of Proposition 7.2. Let us simplify notation and write $X = \mathbf{A}^1_{\mathbf{C}} \smallsetminus S$ and $\mathrm{Gal}(X)$ instead of $\mathrm{Gal}(X)(*)$. We compute $\mathrm{BGal}(X)$ using Lemma 7.3. Note that $\mathrm{sd}(X^{\leq}_{\mathrm{zar}})$ consists of elements of the form

$$\{a\} , \ \{\eta\} , \quad \text{and} \quad \{a < \eta\}$$

for any $a \in \mathbf{C} \smallsetminus S$ and the ordering is given by $\{a\} < \{a < \eta\}$ and $\{\eta\} < \{a < \eta\}$. Furthermore, the functor $\mathrm{Dec}(\mathrm{Gal}(X)) \colon \ \mathrm{sd}(X^{\leq}_{\mathrm{zar}})^{\mathrm{op}} \to \mathbf{Ani}$ can be explicitly described by applying $\widehat{\Pi}^{\text{ét}}_{\infty}$ followed by materialization to the diagram $\mathrm{sd}(X^{\leq}_{\mathrm{zar}})^{\mathrm{op}} \to \mathbf{Sch}$ that sends $\{a\} < \{a < \eta\} > \{\eta\}$ to the span of schemes

$$\begin{tikzcd}
\operatorname{Spec}(\mathbf{C}[T]^{\mathrm{h}}_{(a)}) & \operatorname{Spec}(\mathbf{C}[T]^{\mathrm{h}}_{(a)}) \smallsetminus \{a\} \arrow[l] \arrow[r] & \operatorname{Spec}(\mathbf{C}(T)) ,
\end{tikzcd}$$

see [8, Example 12.2.2]. We for each $a \in \mathbf{C} \smallsetminus S$, we now choose a lift $\bar{\eta}_a$

$$\begin{tikzcd}
 & \operatorname{Spec}(\mathbf{C}[T]^{\mathrm{sh}}_{(a)}) \smallsetminus \{a\} \arrow[d] \\
\operatorname{Spec}(\overline{\mathbf{C}(T)}) \arrow[ur, dotted, "\bar{\eta}_a"] \arrow[r, "\bar{\eta}"'] & \operatorname{Spec}(\mathbf{C}(T))
\end{tikzcd}$$

In particular, we can lift the above span to a span of pointed schemes and therefore also $\mathrm{Dec}(\mathrm{Gal}(X))$ lifts to a diagram of pointed anima $\mathrm{Dec}(\mathrm{Gal}(X))_*$. Using that π_1 is an equivalence between pointed, connected and 1-truncated anima and the category of groups, [HTT, Proposition 7.2.12], we may thus compute

$$\pi_1(\mathrm{BGal}(X), \bar{\eta}) \simeq \operatorname*{colim}_{\mathrm{sd}(X^{\leq}_{\mathrm{zar}})^{\mathrm{op}}} \pi_1(\mathrm{Dec}(\mathrm{Gal}(X))_*)\,.$$

Now for any $\{a\} < \{a < \eta\} > \{\eta\}$, the corresponding span in groups is given by

$$* \longleftarrow \pi_1^{\text{ét}}(\mathrm{Spec}(\mathbf{C}[T]^{\mathrm{sh}}_{(a)}) \smallsetminus \{a\}, \bar{\eta}_a) \longrightarrow \pi_1^{\text{ét}}(\mathrm{Spec}(\mathbf{C}(T)), \bar{\eta})$$

and the colimit over $\mathrm{sd}(X^{\leq}_{\mathrm{zar}})^{\mathrm{op}}$ is given by taking the quotient of $\pi_1^{\text{ét}}(\mathrm{Spec}(\mathbf{C}(T)), \bar{\eta}) = \mathrm{Gal}_{\mathbf{C}(T)}$ by the (abstract) normal closure of the subgroup generated by the images of all the decomposition groups

$$\mathrm{D}_a \coloneqq \pi_1^{\text{ét}}(\mathrm{Spec}(\mathbf{C}[T]^{\mathrm{sh}}_{(a)} \smallsetminus \{a\})\,.$$

By Theorem B.3, there is an isomorphism

$$\widehat{\mathrm{Fr}}_{\mathbf{C}} \xrightarrow{\sim} \mathrm{Gal}_{\mathbf{C}(T)} = \pi_1^{\text{ét}}(\mathrm{Spec}(\mathbf{C}(T)), \bar{\eta})$$

under which the preimage of D_a is, up to conjugation, given by $\widehat{\mathbf{Z}}(a)$. It follows that $\pi_1(\mathrm{BGal}(X), \bar{\eta})$ is isomorphic to the quotient of $\widehat{\mathrm{Fr}}_{\mathbf{C}}$ by the smallest (abstract) normal subgroup containing $\widehat{\mathbf{Z}}(a)$ for all $a \in \mathbf{C} \smallsetminus S$, as desired. ☐

7.4 Corollary. *Let $\bar{x}$ be a geometric generic point of $\mathbf{A}^1_{\mathbf{C}}$. Then*

$$\pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}}, \bar{x}) \neq 1\,.$$

In fact, even the underlying group $\pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}}, \bar{\eta})(*)$ is nontrivial.

Proof. Consider the canonical continuous map $\widehat{\mathrm{Fr}}_{\mathbf{C}} \to \prod_{c \in \mathbf{C}} \widehat{\mathbf{Z}}$ that carries a generator a to the unit vector at a. Note that the (abstract) normal subgroup $N_\varnothing$ lands in the subgroup $\bigoplus_{a \in \mathbf{C}} \widehat{\mathbf{Z}}$. Thus, by Proposition 7.2, we get a short exact sequence of abstract groups

$$\begin{tikzcd}
1 \arrow[r] & N_{\mathbf{C}} \arrow[r] \arrow[d] & \widehat{\mathrm{Fr}}_{\mathbf{C}} \arrow[r] \arrow[d] & \pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}}, \bar{\eta})(*) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \bigoplus_{a \in \mathbf{C}} \widehat{\mathbf{Z}} \arrow[r] & \prod_{a \in \mathbf{C}} \widehat{\mathbf{Z}} \arrow[r, two heads] & Q \arrow[r] & 1,
\end{tikzcd}$$

where $Q \neq 1$ denotes the abstract quotient. The middle vertical map is surjective (because it is dense, the source is profinite and the target is Hausdorff). Therefore, the right vertical map is also surjective. Thus $\pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}}, \bar{\eta})(*) \neq 1$. ☐

The following remark and example, while fitting best the current subsection, use the notion of quasiseparatedness of condensed sets, studied by Clausen–Scholze. The reader might choose to return to them after consulting Subsection 7.2 below, which contains some recollections and further facts about quasiseparated quotients.

7.5 Remark. The proof of Corollary 7.4 can be adapted to show more generally that whenever $\mathbf{C} \smallsetminus S$ is infinite, the condensed group $\pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}} \smallsetminus S, \bar{\eta})$ is not profinite and therefore also not quasiseparated by Theorem 7.17. Indeed, if it was, it would follow from Proposition 7.2 that $N_S \subset \widehat{\mathrm{Fr}}_{\mathbf{C}}$ is a closed subgroup. Thus, the image of N_S under the map $\widehat{\mathrm{Fr}}_{\mathbf{C}} \to \prod_{a\in\mathbf{C}} \widehat{\mathbf{Z}}$ would also be closed in $\prod_{a\in\mathbf{C}} \widehat{\mathbf{Z}}$. But this image is exactly $\bigoplus_{a\in\mathbf{C}\smallsetminus S} \widehat{\mathbf{Z}}$, which is not closed if $\mathbf{C} \smallsetminus S$ is infinite. Even more generally, the above arguments show that the condensed fundamental group of any Dedekind scheme X is not quasiseparated whenever the abstract normal closure $N \subset \mathrm{Gal}_{\mathbf{C}(X)}$ of the subgroup generated by all decomposition groups D is not closed.

The next example shows that whenever $S \neq \varnothing$, even if $\mathbf{C} \smallsetminus S$ is finite, the condensed fundamental group on $\mathbf{A}^1_{\mathbf{C}} \smallsetminus S$ is not *quasiseparated* in the sense of Recollection 7.7. For example, this covers the case of the localization $\mathrm{Spec}(\mathbf{C}[T]_{(T-a)})$ for $a \in \mathbf{C}$.

7.6 Example. Let $G = \widehat{\mathrm{Fr}}_{\{a,b\}}$ be the free profinite group on two elements a, b, and let

$$H = \overline{\langle b \rangle} \simeq \widehat{\mathbf{Z}} \subset G$$

be the (necessarily free) profinite subgroup of G generated by b.

We claim that $H^{\mathrm{nc}} \subsetneq H^{\mathrm{tnc}}$. Indeed, let $g_n = \prod_{i=1}^{n} (a^{i!} b^{i!} a^{-i!})$. For each n, $g_n \in H^{\mathrm{nc}}$. Moreover, the g_n's form a Cauchy net (sequence) in G and thus converge to some $g \in G$, as G is Raĭkov-complete: Indeed, for a given n_0 and $n > n_0$, we have $g_{n_0}^{-1} g_n = \prod_{i=n_0+1}^{n} (a^{i!} b^{i!} a^{-i!})$. Let $N \lhd G$ be a normal open subgroup. Then there exists n_0 such that for any $m \geqslant n_0$, there is $a^{m!}, b^{m!} \in N$. This is because a and b are images of generators of $\widehat{\mathbf{Z}}$ via (two different) continuous maps $\widehat{\mathbf{Z}} \to G$ and the corresponding fact holds already in $\widehat{\mathbf{Z}}$. It now follows that $g_{n_0}^{-1} g_n$ (and, by normality, also $g_n g_{n_0}^{-1}$) lie in N for any $n \geq n_0$. It follows that $g \in H^{\mathrm{tnc}}$. We want to show that $g \notin H^{\mathrm{nc}}$. Assume the contrary. Then there exist some $r \in \mathbf{N}$ and $c_i \in G, d_i \in H$ such that $g = \prod_{i=1}^{r} c_i d_i c_i^{-1}$.

Now, consider the following system of finite quotients of G: $Q_m = P_m \rtimes \mathbf{Z}/m!$, where $P_m = (\mathbf{Z}/m!)^{m!}$ is the $m!$-fold product of $\mathbf{Z}/m!$'s, with an action by $\mathbf{Z}/m!$ that permutes the factors. The map $G \twoheadrightarrow Q_m$ is defined by $b \mapsto (\bar{1}, 0, 0, \ldots) \in P_m = (\mathbf{Z}/m!)^{m!}$ and $a \mapsto \bar{1} \in \mathbf{Z}/m!$. Note that g lands in P_m via this map. Now, for $m \gg r$, we get that, on the one hand, the image of g in $P_m = (\mathbf{Z}/m!)^{m!}$ has an increasing (with m) number of nonzero entries and, on the other hand, the presentation $g = \prod_{i=0}^{r} c_i d_i c_i^{-1}$ implies that this number is bounded by r. This is a contradiction.

Now let $S \subset \mathbf{C}$ be a non-empty subset. We have a diagram of short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & N_S \arrow[r] \arrow[d] & \widehat{\mathrm{Fr}}_{\mathbf{C}} \arrow[r] \arrow[d] & \pi_1^{\mathrm{cond}}(\mathbf{A}^1 \smallsetminus S, \bar{\eta})(*) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & H^{\mathrm{nc}} \arrow[r] & \widehat{\mathrm{Fr}}_{\{a,b\}} \arrow[r] & \widehat{\mathrm{Fr}}_{\{a,b\}}/H^{\mathrm{nc}} \arrow[r] & 1
\end{tikzcd}$$

where the middle vertical map sends $z \in \mathbf{C}$ to b if $z \in S$ and to a otherwise. Then, by construction, H^{nc} is the image of N_S under this map. Thus, if $\pi_1^{\mathrm{cond}}(\mathbf{A}^1 \smallsetminus S, \bar{\eta})$ was quasiseparated, N_S would be a closed subgroup, see Proposition 7.11 below, and so would H^{nc}, contradicting the above.

7.2 Preliminaries on quasiseparated quotients

7.7 Recollection. A condensed set A is *quasiseparated* if for any two maps $B \to A$ and $B' \to A$ in which B and B' are quasicompact, the pullback $B \times_A B'$ is quasicompact as well. We denote by

$\mathrm{Cond}(\mathbf{Set})^{\mathrm{qs}} \subset \mathrm{Cond}(\mathbf{Set})$ the full subcategory that is spanned by the quasiseparated condensed sets.

7.8 Lemma [69, Lemma 4.14]**.** *The inclusion* $\mathrm{Cond}(\mathbf{Set})^{\mathrm{qs}} \subset \mathrm{Cond}(\mathbf{Set})$ *admits a left adjoint* $(-)^{\mathrm{qs}}$ *that preserves finite products.*

Explicitly, if A is a condensed set, its quasiseparated quotient A^{qs} *can be computed by choosing a cover $U = \coprod_i S_i \twoheadrightarrow A$ by profinite sets and by defining A^{qs} as the quotient of U by the closure of the equivalence relation $U \times_A U \subset U \times U$.*

Since $(-)^{\mathrm{qs}}$ preserves finite products, it induces a functor $\mathrm{Cond}(\mathbf{Grp}) \to \mathrm{Cond}(\mathbf{Grp})^{\mathrm{qs}}$ which is left adjoint to the inclusion.

7.9 Notation. Given a scheme X and geometric point $\bar{x} \to X$, we write

$$\pi_1^{\mathrm{cond,qs}}(X, \bar{x}) := \pi_1^{\mathrm{cond}}(X, \bar{x})^{\mathrm{qs}}$$

for the quasiseparated quotient of the condensed fundamental group of X.

Our next goal is to derive a more explicit description of G^{qs}.

7.10 Definition. An inclusion $C \subset A$ of condensed sets is *closed* if for every profinite set S and map $S \to A$, the pullback $C \times_A S \subset S$ is a closed subspace.

7.11 Proposition. *For every condensed group G, its quasiseparated quotient G^{qs} can be computed by $G/\overline{\{e\}}$, where $\overline{\{e\}} \subset G$ is given by the intersection of all closed normal subgroups of G.*

For the proof, we need two auxiliary results.

7.12 Lemma. *Let A be a condensed set and let $R \subset A \times A$ be a closed equivalence relation. Then the quotient A/R is quasiseparated.*

Proof. First, let us choose a cover $U = \coprod_{i \in I} S_i \twoheadrightarrow A$ by profinite sets S_i. Set

$$R_I := R \underset{A \times A}{\times} (U \times U)$$

and note that R_I defines a closed equivalence relation on U with the property that $A/R = U/R_I$. Let Λ be the filtered poset of finite subsets of I, and for each $J \in \Lambda$, let $U_J = \coprod_{j \in J} S_j$. Then we can write U as the filtered union of the U_J, and for each $J \subset J'$ the inclusion $U_J \subset U_{J'}$ is a closed immersion of compact Hausdorff spaces. Let us moreover set

$$R_J := R_I \underset{U \times U}{\times} (U_J \times U_J)$$

for each $J \in \Lambda$. Then each R_J defines a closed equivalence relation on U_J, and, since Λ is filtered, we have $R = \mathrm{colim}_{J \in \Lambda} R_J$. As a consequence, we may identify $A/R = \mathrm{colim}_{J \in \Lambda} U_J/R_J$. Now since each R_J is a closed equivalence relation on U_J, the condensed set U_J/R_J is a compact Hausdorff space. Moreover, for every inclusion $U_J \subset U_{J'}$, the induced map $U_J/R_J \to U_{J'}/R_{J'}$ is injective by construction of R_J and $R_{J'}$ and is therefore automatically a closed immersion. Hence the desired result follows from [69, Proposition 1.2 (4)]. ☐

7.13 Lemma. *Let $\varphi : G \to H$ be a homomorphism of condensed groups. If H is quasiseparated, then* $\ker(\varphi)$ *is a closed subgroup of G.*

Proof. Since $\ker(\varphi)$ is the inverse image of $\{e\} \subset H$, it suffices to show that $\{e\}$ is closed in H. For this, pick any map from a profinite set $S \to H$. Since S and $\{e\}$ are quasicompact and H is quasiseparated, the fiber product $S \times_H \{e\} \subset S$ is quasicompact. Since a subobject of a quasiseparated condensed set is quasiseparated, $S \times_H \{e\}$ is also quasiseparated. It follows that $S \times_H \{e\}$ is compact, and hence a closed subset of S, as desired. ☐

Proof of Proposition 7.11. We begin by showing that the quotient $G/\overline{\{e\}}$ is quasiseparated. To see this, first note that the map

$$(\mathrm{pr}_0, \mu) \colon G \times \overline{\{e\}} \to G \times G \tag{7.14}$$

(where μ denotes the multiplication map) is a closed immersion since when composing this map with the isomorphism $G \times G \to G \times G$ given by $(g, h) \mapsto (g, g^{-1}h)$, the resulting map can be identified with the product of the identity with the inclusion. Observe that the map in (7.14) is precisely the equivalence relation defining the quotient group $G/\overline{\{e\}}$. Hence the claim follows from Lemma 7.12.

To complete the proof, we need to show that for every map $\varphi \colon G \to H$ of condensed groups in which H is quasiseparated, the kernel $\ker(\varphi)$ contains $\overline{\{e\}}$. For this, it suffices to check that $\ker(\varphi)$ is closed. This is Lemma 7.13. ☐

7.15 Proposition. *Let* $1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1$ *be a short exact sequence of condensed groups. If* H *is quasiseparated, the induced sequence* $1 \longrightarrow N^{\mathrm{qs}} \longrightarrow G^{\mathrm{qs}} \longrightarrow H \longrightarrow 1$ *is again exact.*

Proof. We only need to show that $N^{\mathrm{qs}} \to G^{\mathrm{qs}}$ is injective (we are using here $H = H^{\mathrm{qs}}$). Since H is quasiseparated, Lemma 7.13 shows that $N \to G$ is closed. Therefore, $\overline{\{1\}}^N = \overline{\{1\}}^G$ (as subgroups of G), and thus

$$N^{\mathrm{qs}} = N/\overline{\{1\}}^N \longrightarrow G/\overline{\{1\}}^G = G^{\mathrm{qs}}$$

is injective. ☐

7.16 Corollary (fundamental exact sequence on quasiseparated quotients)**.** *Let* k *be a field with separable closure* $\bar{k}$*, let* X *be a qcqs* k*-scheme, and let* $\bar{x} \to X_{\bar{k}}$ *be a geometric point. If* X *is geometrically connected and* $X_{\bar{k}}$ *has finitely many irreducible components, then the sequence of condensed groups*

$$1 \longrightarrow \pi_1^{\mathrm{cond,qs}}(X_{\bar{k}}, \bar{x}) \longrightarrow \pi_1^{\mathrm{cond,qs}}(X, \bar{x}) \longrightarrow \mathrm{Gal}_k \longrightarrow 1$$

is exact.

Proof. Combine Corollary 5.7 and Remark 5.8 with Proposition 7.15. ☐

7.3 $\pi_1^{\mathrm{cond,qs}}$ of geometrically unibranch schemes

It is a common theme in arithmetic geometry that various generalizations of $\pi_1^{\text{ét}}$ are all equal (and profinite) for normal (more generally: geometrically unibranch) schemes. See [7, Theorem 11.1] and [10, Lemma 7.4.10] for instances of this phenomenon. As we saw before, this fails for π_1^{cond} and $X = \mathbf{A}^1_{\mathbf{C}}$. However, the expected behavior still holds for $\pi_1^{\mathrm{cond,qs}}$. Proving this fact is the main goal of this subsection.

7.17 Theorem. *Let X be a qcqs, geometrically unibranch scheme with finitely many irreducible components, and let $\bar{x}$ be a geometric point of X. Then the natural homomorphism $\pi_1^{\mathrm{cond}}(X,\bar{x}) \to \pi_1^{\mathrm{\acute{e}t}}(X,\bar{x})$ induces an isomorphism*

$$\pi_1^{\mathrm{cond,qs}}(X,\bar{x}) \xrightarrow{\sim} \pi_1^{\mathrm{\acute{e}t}}(X,\bar{x})\,.$$

In particular, $\pi_1^{\mathrm{cond,qs}}(X,\bar{x})$ is a profinite group.

For the proof, we need the following observation.

7.18 Proposition. *Let X be a qcqs scheme with a geometric point $\bar{x}$ and such that $\pi_0^{\mathrm{cond}}(X)$ is discrete. Then the canonical comparison homomorphism*

$$\pi_1^{\mathrm{cond}}(X,\bar{x}) \to \pi_1^{\mathrm{\acute{e}t}}(X,\bar{x})$$

witnesses $\pi_1^{\mathrm{\acute{e}t}}(X,\bar{x})$ as the profinite completion of $\pi_1^{\mathrm{cond}}(X,\bar{x})$. The condition on π_0^{cond} is satisfied (for example) when X has locally finitely many irreducible components.

Proof. Combine Lemma 2.12, Lemma 3.14, and Corollary 4.19. ☐

To prove the main result, we first want to show that this quasiseparated quotient is a compact topological group. For this, we make use of the following simple consequence of the fact that the fundamental group of a simplicial set coincides with the fundamental group of its geometric realization:

7.19 Lemma. *Let $f : T_\bullet \to S_\bullet$ be a map of simplicial sets that is bijective on vertices and surjective on edges. Then, for any choice of basepoint $t \in T_0$, the induced homomorphism*

$$f_* : \pi_1(T_\bullet, t) \to \pi_1(S_\bullet, f(t))$$

is surjective. ☐

7.20 Lemma. *Let $Y \to X$ be a morphism of qcqs schemes. Assume that there exist proétale hypercovers $X'_\bullet \to X$ and $Y'_\bullet \to Y$ by w-strictly local schemes and a morphism $Y'_\bullet \to X'_\bullet$ that fit into a commutative diagram*

$$\begin{tikzcd}
Y'_\bullet \arrow[r] \arrow[d] & X'_\bullet \arrow[d] \\
Y \arrow[r] & X
\end{tikzcd}$$

such that:

(1) *The induced map of profinite sets $\pi_0(Y'_0) \to \pi_0(X'_0)$ is a bijection (and thus, a homeomorphism).*

(2) *The induced map of profinite sets $\pi_0(Y'_1) \to \pi_0(X'_1)$ is a surjection (and thus, a topological quotient map).*

Then, for any choice of geometric points $\bar{y} \mapsto \bar{x}$, the induced homomorphism

$$\pi_1^{\mathrm{cond}}(Y,\bar{y}) \to \pi_1^{\mathrm{cond}}(X,\bar{x})$$

is a surjection of condensed groups.

Proof. By Recollection 2.7 and Propositions 3.16 and 3.40, the fundamental group $\pi_1^{\mathrm{cond}}(X, \bar{x})$ can be computed as

$$\mathbf{Extr}^{\mathrm{op}} \ni S \mapsto \pi_1\Big(\operatorname*{colim}_{[m] \in \boldsymbol{\Delta}^{\mathrm{op}}} \mathrm{Map}_{\mathbf{Top}}(S, \pi_0(X'_m)), \bar{x} \Big) .$$

In other words, for each S, we have to compute the fundamental group of the simplicial set $\mathrm{Map}_{\mathbf{Top}}(S, \pi_0(X'_\bullet))$ given by $[m] \mapsto \mathrm{Map}_{\mathbf{Top}}(S, \pi_0(X'_m))$. Analogous statements hold for $Y'_\bullet$ and Y.

Now, the assumptions on the maps $\pi_0(Y'_0) \to \pi_0(X'_0)$ and $\pi_0(Y'_1) \to \pi_0(X'_1)$ imply that, for each $S \in \mathbf{Extr}$, the induced map

$$\mathrm{Map}_{\mathbf{Top}}(S, \pi_0(Y'_\bullet)) \to \mathrm{Map}_{\mathbf{Top}}(S, \pi_0(X'_\bullet))$$

of simplicial sets satisfies the assumptions of Lemma 7.19. It follows that, for each S, the map

$$\pi_1^{\mathrm{cond}}(Y, \bar{y})(S) \to \pi_1^{\mathrm{cond}}(X, \bar{x})(S)$$

is a surjection, as desired. ☐

7.21 Lemma. *Let X be a quasiseparated, geometrically unibranch, irreducible scheme and let $\eta \in X$ be its generic point. Let $X_\bullet$ be any proétale hypercover by w-contractible qcqs schemes of X. Then there exists a proétale hypercover $Y_\bullet$ of η satisfying the conditions of Lemma 7.20 (with respect to $X_\bullet$ and the map $\eta \to X$).*

Proof. Let $X_{\bullet,\eta}$ be the basechange of $X_\bullet$ to η and note that the map $\pi_0(X_{\bullet,\eta}) \to \pi_0(X_\bullet)$ is a levelwise homeomorphism by geometrical unibranchness and the fact that each connected component of a w-contractible proétale X' over X is the strict localization at some geometric point of X (see e.g. [51, Lemma 3.15]). In particular, the profinite sets $\pi_0(X_{i,\eta})$ are still extremally disconnected. Being w-strictly local, however, will usually be lost after base-changing to η. We want to define a w-strictly local hypercover $W_\bullet$ of η with a map to $X_{\bullet,\eta}$ that still has the desired properties on π_0 in low degrees.

To do that, fix a geometric point $\bar{\eta}$ lying over η and write $X_{0,\bar{\eta}} := X_{0,\eta} \times_\eta \bar{\eta}$. The projection induces a surjective map of profinite sets $\pi_0(X_{0,\bar{\eta}}) \to \pi_0(X_{0,\eta})$. As the target is extremally disconnected, this map admits a section. Let $T \subset \pi_0(X_{0,\bar{\eta}})$ be the image of one such section. By [10, Lemma 2.2.8], there exists a pro-(Zariski localization) $W_0 \to X_{0,\bar{\eta}}$ that realizes the map $T \subset \pi_0(X_{0,\bar{\eta}})$ on connected components. Such W_0 will in particular be weakly étale over $\bar{\eta}$, so w-strictly local by Example 2.40, and, by construction, the map $\pi_0(W_0) \to \pi_0(X_{0,\eta})$ induced by $W_0 \to X_{0,\bar{\eta}} \to X_{0,\eta}$ is a homeomorphism. We can extend this to a map of hypercovers

$$Y_\bullet := \mathrm{cosk}_0(W_0) \underset{\mathrm{cosk}_0(X_{\bullet,\eta})}{\times} X_{\bullet,\eta} \to X_{\bullet,\eta}$$

that induces a bijection on 0-simplices. The map on 1-simplices is explicitly given by

$$(W_0 \times_\eta W_0) \underset{X_{0,\eta} \times_\eta X_{0,\eta}}{\times} X_{1,\eta} \to X_{1,\eta}$$

and is therefore surjective since $W_0 \to X_{0,\eta}$ is surjective. Furthermore, all terms of $Y_\bullet$ are w-strictly local since they are all weakly étale over $\bar{\eta}$ Example 2.40. This completes the proof. ☐

7.22 Corollary. *Let X be a quasiseparated, geometrically unibranch, irreducible scheme with generic point $\eta \in X$. Choose a geometric point $\bar{\eta}$ lying over η. Then the canonical map*

$$\mathrm{Gal}_{\kappa(\eta)} = \pi_1^{\mathrm{cond}}(\mathrm{Spec}(\kappa(\eta)), \bar{\eta}) \twoheadrightarrow \pi_1^{\mathrm{cond}}(X, \bar{\eta})$$

is a surjection of condensed groups.

Proof. Combine Lemmas 7.20 and 7.21 and Example 3.38. ☐

7.23 Lemma. *Let $G' \twoheadrightarrow G$ be a surjection of condensed groups. Assume that G' is a profinite group. Then the quasiseparated quotient G^{qs} is a profinite group.*

Proof. The quotient G^{qs} is qcqs (it is qc as a quotient of something qc). By [17, Proposition 2.8], its underlying condensed set is a compact topological space. It follows (as the embedding of compact(ly generated) spaces into condensed sets is fully faithful and commutes with products) that G^{qs} is a (Hausdorff) compact group admitting a surjection from a profinite group G'. Hence G^{qs} is itself profinite. ☐

Finally, we are ready to prove the main result of this subsection.

Proof of Theorem 7.17. Note that, since $\mathrm{Pro}(\mathbf{Grp}_{\mathrm{fin}}) \subset \mathrm{Cond}(\mathbf{Grp})^{\mathrm{qs}} \subset \mathrm{Cond}(\mathbf{Grp})$, the profinite completion $G^{\wedge}$ of a condensed group G factors over the quasiseparated quotient G^{qs} of G. Our assumptions guarantee that every connected component is irreducible. By the preceding preparatory results Corollary 7.22 and Lemma 7.23, we thus have that $\pi_1^{\mathrm{cond,qs}}(X, \bar{x})$ is already profinite, hence agrees with the profinite completion $\pi_1^{\mathrm{cond}}(X, \bar{x})^{\wedge}$. By Proposition 7.18, this latter profinite completion recovers $\pi_1^{\mathrm{\acute{e}t}}(X, \bar{x})$. This completes the proof. ☐

7.24 Remark. It seems like a natural idea to try to extend the notion of quasiseparatedness and quasiseparated quotients to all *condensed anima*, and also extend Theorem 7.17 from fundamental groups to homotopy types. However, a sufficiently nicely behaved quasiseparated quotient of condensed anima can *not* exist. More precisely, there is *no* full subcategory $\mathcal{C} \subset \mathrm{Cond}(\mathbf{Ani})$ with the following properties:

(1) The inclusion $\mathcal{C} \subset \mathrm{Cond}(\mathbf{Ani})$ admits a left adjoint $(-)^{\mathrm{qs}}$.

(2) A condensed set is in $\mathcal{C}$ if and only if its is quasiseparated.

(3) For any quasiseparated condensed group G, the condensed anima $\mathrm{B}G$ is contained in $\mathcal{C}$.

Indeed, both $\mathrm{B}\mathbf{Z}$ and $\mathrm{B}\widehat{\mathbf{Z}}$ would be contained in $\mathcal{C}$. Since $\widehat{\mathbf{Z}}/\mathbf{Z}$ is the fiber of the canonical map $\mathrm{B}\mathbf{Z} \to \mathrm{B}\widehat{\mathbf{Z}}$, the condensed set $\widehat{\mathbf{Z}}/\mathbf{Z}$ would also be contained in $\mathcal{C}$. But $\widehat{\mathbf{Z}}/\mathbf{Z}$ is not quasiseparated.

7.4 The van Kampen and Künneth formulas for $\pi_1^{\mathrm{cond,qs}}$

Let us first define a free (nonabelian!) condensed group on a compact space T (or, more generally, condensed set M). The forgetful functor $\iota\colon \mathrm{Cond}(\mathbf{Grp}) \to \mathrm{Cond}(\mathbf{Set})$ has a left adjoint

$$\mathrm{Fr}_{(-)}^{\mathrm{cond}}\colon \mathrm{Cond}(\mathbf{Set}) \to \mathrm{Cond}(\mathbf{Grp}).$$

The condensed group $\mathrm{Fr}_M^{\mathrm{cond}}$ is given more explicitly as the sheafification of the functor

$$\begin{aligned} \mathrm{Fr}_M^{\mathrm{pre}} : \ \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})^{\mathrm{op}} &\to \mathbf{Grp} \\ S &\mapsto \mathrm{Fr}_{M(S)} \ . \end{aligned}$$

The free group on M comes with a canonical map $M \to \mathrm{Fr}_M^{\mathrm{cond}}$ in Cond(**Set**). For a profinite set T, we want to compare it with $\mathrm{Fr}_T^{\mathrm{top}}$, i.e., the free topological group on T (see [5, Chapter 7]). Note that, by the universal property of $\mathrm{Fr}_T^{\mathrm{cond}}$, there is a canonical homomorphism

$$\mathrm{Fr}_T^{\mathrm{cond}} \to \underline{\mathrm{Fr}_T^{\mathrm{top}}}$$

in Cond(**Grp**).

7.25 Recollection (on free topological groups and products)**.** In this recollection, T will always denote a topological space and G_i will denote topogical groups.

(1) (Markov, c.f. [5, Theorems 7.1.2 & 7.1.5]) The free topological group $\mathrm{Fr}_T^{\mathrm{top}}$ on T exists for every Tychonoff (=completely regular) space T, and the unit $\eta : T \to \mathrm{Fr}_T^{\mathrm{top}}$ is a topological embedding. In addition, the image $\eta(T)$ is a free algebraic basis for G.

(2) (Graev, Mack, Morris, Ordman, c.f. [5, Theorem 7.4.1]) When T is compact (more generally: k_ω), then $\mathrm{Fr}_T^{\mathrm{top}}$ is the topological colimit of subspaces

$$(\mathrm{Fr}_T)_{\leq n} = B_n(T) = \{\text{words of reduced length } \leq n\} \, .$$

(3) By [31], the underlying set of $*_i^{\mathrm{top}} G_i$ is the abstract free product and if the groups are Hausdorff, their free product is Hausdorff too.

Moreover (c.f. [52, Remark 4.27]), when the G_i's are either compact or finitely generated discrete (say $\mathbf{Z}^{*r}$), by looking at the surjection from a suitable free product (see Lemma 7.30 below) and using (1), it follows that $*_i^{\mathrm{top}} G_i$ is a topological colimit of compact subsets of "bounded words", where by "bounded words" we in particular mean that all "letters" from one of the **Z**'s sit inside of some (larger and larger) interval $[-n, n]$.

7.26 Recollection. In the context of (abstract) free groups on a set M (resp., free products of groups $G_1, \dots, G_n$) we say that $g_{m_1}^{r_1} \cdots g_{m_n}^{r_n}$ (resp., $g_1 \dots g_n$), where g_{m_i} is the generator corresponding to $m_i \in M$ (resp., where g_i is a nontrivial element of one of the groups G_j, say $G_{j(i)}$) is a *reduced word* if for $1 \leq i < n$, there is $m_i \neq m_{i+1}$ (resp., $j(i) \neq j(i+1)$).

The following result is a nonabelian analogue of [69, Proposition 2.1]. The proof essentially follows the one of *loc. cit.*

7.27 Proposition. *Let T be a compact Hausdorff topological space. Then the natural map*

$$\mathrm{Fr}_T^{\mathrm{cond}} \to \underline{\mathrm{Fr}_T^{\mathrm{top}}} \tag{7.28}$$

is an isomorphism.

In the proof, we use the following convention: For a profinite set S and $t \in T(S)$, we denote by $g_t \in \mathrm{Fr}_T^{\mathrm{cond}}$ the element given by the composition

$$S \xrightarrow{t} T \to \mathrm{Fr}_T^{\mathrm{cond}} \ ,$$

where $T \to \mathrm{Fr}_T^{\mathrm{cond}}$ is the unit map.

Proof. First, we want to check that the map (7.28) is injective. Note that this boils down to checking that any section of $\mathrm{Fr}^{\mathrm{pre}}_T$ that maps to $1 \in \underline{\mathrm{Fr}^{\mathrm{top}}_T}$, trivializes after passing to a cover in $\mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$.

Observe that this is the case for the underlying groups. Indeed, it is enough to check that the map $\mathrm{Fr}_{T(*)} \to \mathrm{Fr}^{\mathrm{top}}_T(*)$ is injective.[6] This follows directly from Recollection 7.25 (1).

We now treat the injectivity for a general $S \in \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$. Assume that $1 \neq g \in \mathrm{Fr}_{T(S)}$ maps to $1 \in \mathrm{Fr}^{\mathrm{top}}_T(S)$. By the previous point, for any $s \in S$, the restriction $g(s) \in \mathrm{Fr}_{T(*)}$ is trivial. Write g as a reduced word $g = g_{t_1}^{r_1} g_{t_2}^{r_2} \cdots g_{t_m}^{r_m}$, where now $t_j \in T(S)$. All g_{t_j} are nonzero and, if $m > 1$, we have $g_{t_i} \neq g_{t_{i+1}}$ for $1 \le i \le m-1$.

If $m = 1$, then we plug in any $s \in S$ to see that $1 = g(s) = g_{t_1(s)}^{r_1}$. But the right hand side cannot be trivial being a generator in the free group raised to a nonzero power – a contradiction.

Assume now that $m > 1$. Let S_j denote the closed subset of S where $t_j = t_{j+1}$. First, note that the S_j's (where $1 \le j < m$) jointly cover S. Indeed, if that would not be the case, then any point s in the complement would have the property that $1 = g(s) = g_{t_1(s)}^{r_1} g_{t_2(s)}^{r_2} \cdots g_{t_m(s)}^{r_m}$ is a nontrivial reduced word – a contradiction.

Thus, passing to a finite closed cover of S, we can assume that $t_j = t_{j+1}$ for some j, effectively reducing the length of the shortest word that g can be written as. By induction, this implies that g has to be trivial – a contradiction.

As the proof of injectivity is finished, we now move on to surjectivity. Consider the map of compact topological spaces

$$T^n \times \{-1,0,1\}^n \to (\mathrm{Fr}^{\mathrm{top}}_T)_{\le n}$$

given by $(t_1, \dots, t_n, \varepsilon_1, \dots, \varepsilon_n) \mapsto g_{t_1}^{\varepsilon_1} \cdots g_{t_n}^{\varepsilon_n}$. This map is clearly surjective. It fits into a commutative square

$$\begin{tikzcd}
T^n \times \{-1,0,1\}^n \arrow[r] \arrow[d] & (\mathrm{Fr}^{\mathrm{top}}_T)_{\le n} \arrow[d] \\
\mathrm{Fr}^{\mathrm{cond}}_T \arrow[r] & \bigcup_m (\mathrm{Fr}^{\mathrm{top}}_T)_{\le m} = \mathrm{Fr}^{\mathrm{top}}_T .
\end{tikzcd}$$

Evaluating at any $S \in \mathbf{Extr}$, and using [10, Lemma 4.3.7], this shows the surjectivity of the lower horizontal map (by varying n). ☐

7.29 Remark. Assume that $S = \lim_i S_i$ is a profinite set with S_i finite. Essentially, the same proof strategy (but without having to use the results of Recollection 7.25 (1)) shows further that $\mathrm{Fr}^{\mathrm{cond}}_S$ and $\underline{\mathrm{Fr}^{\mathrm{top}}_S}$ are isomorphic to the group $\underline{\bigcup_m \lim_i \left((\mathrm{Fr}_{S_i})_{\le m}\right)}$. This is analogous to the presentation in [69, Proposition 2.1].

Similarly as before, one can introduce the free condensed product $*^{\mathrm{cond}}$ of condensed groups. It is the coproduct in the category of condensed groups. It can be explicitly described as the sheafification of the presheaf $*^{\mathrm{pre}}_i\, G_i$ given by

$$\begin{aligned}\mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})^{\mathrm{op}} &\to \mathbf{Grp}\\ S &\mapsto *_i\, G_i(S)\,.\end{aligned}$$

---

[6] We are using here that evaluating $\mathrm{Fr}^{\mathrm{cond}}_W$ on $*$ as a sheaf is the same as evaluating its defining presheaf.

Free products of topological groups $*^{\mathrm{top}}$ exist as well and for $G_i \in \mathrm{Grp}(\mathbf{Top})$ there is a canonical homomorphism $*_i^{\mathrm{cond}} \underline{G_i} \to \underline{*_i^{\mathrm{top}} G_i}$.

First, let use prove an auxiliary lemma.

7.30 Lemma. *Let $G_1, \dots, G_m$ be compact Hausdorff topological groups and $r \in \mathbf{N}$. Denote by $T = G_1 \sqcup \cdots \sqcup G_m \sqcup \{1, \dots, r\}$ the topological space that is the disjoint union of the G_i's and r singletons. Then the canonical homomorphism*

$$\mathrm{Fr}_T^{\mathrm{cond}} \to G_1 *^{\mathrm{cond}} \cdots *^{\mathrm{cond}} G_m *^{\mathrm{cond}} \mathbf{Z}^{*^{\mathrm{cond}} r}.$$

is surjective. An analogous fact holds for topological free products.

Proof. The universal properties of these groups give a homomorphism as above (here, we are mapping each of the r points in T to $1 \in \mathbf{Z}$ via one of the r canonical maps $\mathbf{Z} \to \mathbf{Z}^{*^{\mathrm{cond}} r}$). This map already exists on the level of the defining presheaves and is surjective there, so the map of sheaves is surjective as well.

We omit the details for the topological counterpart (it uses Recollection 7.25). ☐

7.31 Proposition. *Let $G_1, \dots, G_m$ be compact Hausdorff topological groups and $r \in \mathbf{N}$. Then the natural map*

$$G_1 *^{\mathrm{cond}} \cdots *^{\mathrm{cond}} G_m *^{\mathrm{cond}} \mathbf{Z}^{*^{\mathrm{cond}} r} \to \underline{G_1 *^{\mathrm{top}} \cdots *^{\mathrm{top}} G_m *^{\mathrm{top}} \mathbf{Z}^{*^{\mathrm{top}} r}}$$

is an isomorphism in $\mathrm{Cond}(\mathbf{Grp})$.

Proof. To see the surjectivity, one can either redo the argument in the proof of Proposition 7.27 or use its statement together with Lemma 7.30 and the diagram (with $T = G_1 \sqcup \dots \sqcup G_m \sqcup * \sqcup \dots \sqcup *$)

$$\begin{tikzcd}
\mathrm{Fr}_T^{\mathrm{cond}} \arrow[r] \arrow[d] & *_i^{\mathrm{cond}} G_i \arrow[d] \\
\underline{\mathrm{Fr}_T^{\mathrm{top}}} \arrow[r] & \underline{*_i^{\mathrm{top}} G_i}
\end{tikzcd}$$

Now, for the injectivity, the argument is very similar to the proof of Proposition 7.27. We can work with $*_i^{\mathrm{pre}} G_i$. The homomorphism of underlying groups

$$*_i G_i(*) \to \left(*_i^{\mathrm{top}} G_i \right)(*)$$

is a bijection (see Recollection 7.25).

Now, fix $S \in \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$ and let $g = g_1 g_2 \cdots g_n \in *_i G_i(S)$ be mapping to $1 \in \left(*_i^{\mathrm{top}} G_i \right)(S)$. Here, each g_j is in some $G_{\alpha(j)}(S)$ and we can assume this presentation of g is a reduced word (we assume $m > 1$ as the case when $m = 1$ is again easy). We know that $\mathrm{g}(s) \in *_j G_j(*)$ is trivial for any $s \in S$.

Let S_j denote the closed subsets of S where g_j vanishes. First, note that the S_j's (where $1 \leq j \leq n$) jointly cover S. Indeed, if that's not the case, then any point s in the complement would have the property that $g(s) = g_1(s) g_2(s) \cdots g_n(s)$ is a nontrivial reduced word – a contradiction.

But now, passing to the this cover, we have again reduced the length of the presentation of g as a word. We are done by induction. ☐

7.32 Lemma. *Let T be a compactly generated topological space. Sending a closed subspace $Z \subset T$ to $\underline{Z} \to \underline{T}$ induces an order-preserving bijection between closed subspaces of T and closed condensed subsets of $\underline{T}$. The inverse is given by sending a closed condensed subset $Z \subset \underline{T}$ to $Z(*) \subset \underline{T}(*) = T$ equipped with the subspace topology.*

Proof. In order to avoid confusion during the proof, we will write $\underline{S}$ for the condensed set represented by a profinite set S. We at first check that the inverse defined above is well-defined, that is, that $Z(*)$ is a closed subset of T. We may check this after pulling back along any continuous map $f : S \to T$ for S a profinite set. Then the pullback $S \times_T Z(*) \subset S$ is the subspace given by those $s \in S$ such that $f(s) \in Z(*)$. If we alternatively compute the pullback $Z \times_{\underline{T}} \underline{S}$ in $\mathrm{Cond}(\mathbf{Set})$, then $Z \times_{\underline{T}} \underline{S} \subset \underline{S}$ is a closed condensed subset by definition. In particular, $(Z \times_{\underline{T}} \underline{S})(*)$ is a closed subset of S. But $(Z \times_{\underline{T}} \underline{S})(*) = Z(*) \times_T S$, as subsets of S, and thus $Z(*)$ is closed.

Furthermore, for a closed subspace $Z \subset T$, we have $Z = \underline{Z}(*)$. So, conversely, let us start with a closed condensed subset $Z \subset \underline{T}$. Then for any $S \in \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$ we claim that the subset $Z(S) \subseteq T(S)$ is given by those $f : \underline{S} \to \underline{T}$ such that for all $s \in S$, $f(s) \in Z(*)$. Indeed, since Z is a subobject, f is in $Z(S)$, if and only if the monomorphism $j : Z \times_{\underline{T}} \underline{S} \to \underline{S}$ is an isomorphism. But since j is a closed immersion, it follows that j is an isomorphism if and only if $j(*)$ is. But this is the case if and only $f(s) \in Z(*)$ for all $s \in S$, as claimed. Since the same description applies to the condensed subset represented by the subspace $Z(*)$ equipped with the closed subspace structure, the claim follows. □

7.33 Corollary. *Let G be a topological group and $H \lhd \underline{G}$ a normal condensed subgroup. Assume that G^{qs} is represented by a compactly generated topological group G_0. Let $H_0 = \mathrm{im}(H \to G \to G^{\mathrm{qs}} = \underline{G_0})$. Then the canonical homomorphism*

$$(G/H)^{\mathrm{qs}} \xrightarrow{\sim} \underline{G_0 \big/ \overline{H_0(*)}} \quad \textit{in } \mathrm{Cond}(\mathbf{Grp})$$

is an isomorphism, where $\overline{H_0(*)}$ denotes the topological closure in G.

Proof. Comparing universal properties, we see that the canonical map $(G/H)^{\mathrm{qs}} \to (G^{\mathrm{qs}}/H_0)^{\mathrm{qs}}$ is an isomorphism. By Proposition 7.11, it follows further that $(G^{\mathrm{qs}}/H_0)^{\mathrm{qs}} \to G^{\mathrm{qs}}/\overline{H_0}$. Now since $G^{\mathrm{qs}} = \underline{G_0}$, Lemma 7.32 shows that $\overline{H_0} = \overline{H_0(*)}$, completing the proof. □

We now fix some notation for the van Kampen formula.

7.34 Notation. Let X be a scheme.

(1) Assume X is connected and has finitely many irreducible components. Write $\nu : X^{\nu} \to X$ for the normalization and write

$$X^{2\nu} \coloneqq X^{\nu} \times_X X^{\nu} \qquad \text{and} \qquad X^{3\nu} \coloneqq X^{\nu} \times_X X^{\nu} \times_X X^{\nu} .$$

Assume that $X^{2\nu}$ and $X^{3\nu}$ also have finitely many irreducible components (e.g., when X is Nagata). Decompose $X^{\nu} = \coprod_i X_i^{\nu}$ into connected components. Write Γ for the "dual" graph with vertices $V = \pi_0(X^{\nu})$ and edges $E = \pi_0(X^{2\nu})$, and fix a maximal tree T.

(2) We write $\Pi_1^{\mathrm{cond}}(X) \coloneqq \tau_{\leq 1}\Pi_{\infty}^{\mathrm{cond}}(X)$ (resp. $\widehat{\Pi}_1^{\mathrm{\acute{e}t}}(X) \coloneqq \tau_{\leq 1}\widehat{\Pi}_{\infty}^{\mathrm{\acute{e}t}}(X)$) for the *condensed (resp. profinite étale) fundamental groupoid* of X. Here, $\tau_{\leq 1}$ denotes 1-truncation of condensed (resp. profinite) anima.

7.35 Theorem (van Kampen formula for the quasiseparated fundamental group)**.** *In the notation of Notation 7.34, after making choices of geometric base points and étale paths (as in [73, Corollary 5.3]), one has a canonical isomorphism*

$$\pi_1^{\mathrm{cond,qs}}(X,\bar{x}) \simeq \underline{\big(*_i^{\mathrm{top}}\ \pi_1^{\text{ét}}(X_i^{\nu},\bar{x}_i) *^{\mathrm{top}} \pi_1(\Gamma,T)\big)/H^{\mathrm{tnc}}}\ ,$$

where H is the subgroup generated by the following relations:

(1) *For all $e \in E$ and $g \in \pi_1^{\text{ét}}(e,\bar{x}(e))$ we have $\pi_1^{\text{ét}}(\partial_1)(g)\vec{e} = \vec{e}\pi_1^{\text{ét}}(\partial_0)(g)$.*

(2) *For all $f \in \pi_0(X^{3\nu})$, we have*

$$\overrightarrow{(\partial_2 f)}\alpha^{(f)}_{102}(\alpha^{(f)}_{120})^{-1}\overrightarrow{(\partial_0 f)}\alpha^{(f)}_{210}(\alpha^{(f)}_{201})^{-1}\Big(\overrightarrow{(\partial_1 f)}\Big)^{-1}\alpha^{(f)}_{021}(\alpha^{(f)}_{012})^{-1} = 1\ .$$

Here, each $\alpha^{(f)}_{ijk}$ lives in some $\pi_1^{\text{ét}}(X_l^{\nu},\bar{x}_l)$ and $\vec{e}, \overrightarrow{(\partial_i f)} \in \pi_1(\Gamma,T)$.

Proof. Combining Corollary 6.16, left adjointness of 1-truncation and [41, Proposition A.1], we get an equivalence of condensed groupoids.

$$\operatorname*{colim}_{[k]\in\mathbf{\Delta}^{\mathrm{op}}_{\leqslant 2}} \Pi_1^{\mathrm{cond}}(X^{k\nu}) \xrightarrow{\sim} \Pi_1^{\mathrm{cond}}(X).$$

The fixed geometric points and étale paths fix points and paths in $\Pi_1^{\mathrm{cond}}(X)(*), \Pi_1^{\mathrm{cond}}(X_i^{\nu})(*), \ldots$ and so in any $\Pi_1^{\mathrm{cond}}(X)(S), \Pi_1^{\mathrm{cond}}(X_i^{\nu})(S), \ldots$ for $S \in \mathbf{Extr}$. By Corollary 4.19, these groupoids are connected. We now want to pass from a statement about fundamental groupoids to a statement involving fundamental groups. For a fixed $S \in \mathbf{Extr}$, we can apply the usual "discrete" van Kampen formula: see [52, Theorem 3.7] for a version for 2-complexes of Noohi (and so also discrete) groups or [11, Chapter IV, §5], cf. also [73]. It implies that

$$\pi_1^{\mathrm{cond}}(X,\bar{x}) \simeq \big(*_i^{\mathrm{cond}}\ \pi_1^{\mathrm{cond}}(X_i^{\nu},\bar{x}_i) *^{\mathrm{cond}} \pi_1(\Gamma,T)\big)/H'$$

where H' is the normal condensed subgroup that for each S is generated by relations analogous relations as in the statement, but where $g \in \pi_1^{\mathrm{cond}}(e,\bar{x}(e))(S)$, etc.

Now, passing to quasiseparated quotients and using $\pi_1^{\mathrm{cond}}(X_i^{\nu},\bar{x}_i)^{\mathrm{qs}} = \pi_1^{\text{ét}}(X_i^{\nu},\bar{x}_i)$ (this is Theorem 7.17) together with Proposition 7.31 and Corollary 7.33 yields the result.

We have used the following observation to get $g \in \pi_1^{\text{ét}}(e,\bar{x}(e))$ as opposed to $g \in \pi_1^{\mathrm{cond,qs}}(e,\bar{x}(e))$ or $g \in \pi_1^{\mathrm{cond}}(e,\bar{x}(e))$ in relation (1): although $X^{2\nu}$ might not be normal, so $\pi_1^{\mathrm{cond,qs}}(e,\bar{x}(e))$ might differ from $\pi_1^{\text{ét}}(e,\bar{x}(e))$, the maps $\pi_1^{\mathrm{cond,qs}}(\partial_1), \pi_1^{\mathrm{cond,qs}}(\partial_0)$ have profinite groups as the targets and thus, factorize through the profinite completion of $\pi_1^{\mathrm{cond,qs}}(e,\bar{x}(e))$, which is $\pi_1^{\text{ét}}(e,\bar{x}(e))$ (cf. Proposition 7.18). As the topological normal closure of the image of $\pi_1^{\mathrm{cond,qs}}(e,\bar{x}(e))(*)$ inside $\pi_1^{\text{ét}}(e,\bar{x}(e)$ is the whole group (one uses the universal property of the profinite completion to check this), the set of relations

$$\{\pi_1^{\text{ét}}(\partial_1)(g)\vec{e}\pi_1^{\text{ét}}(\partial_0)(g)^{-1}\vec{e}^{-1} | e \in E, g \in \pi_1^{\text{ét}}(e,\bar{x}(e))\}$$

is still in H^{tnc} and contains the original set of relations (i.e. a similarly defined one where $g \in \pi_1^{\mathrm{cond,qs}}(e,\bar{x}(e))$), as desired. □

7.36 Example. Let k be a separably closed field.

(1) Let C_1 and C_2 be normal curves over k with fixed closed points $c_i \in C_i$. Let $C = C \sqcup_{c_1=c_2} C_2$ be the gluing of these curves along these closed points. Then

$$\pi_1^{\mathrm{cond,qs}}(C, c) \simeq \pi_1^{\text{ét}}(C_1, c_1) *^{\mathrm{top}} \pi_1^{\text{ét}}(X_2, c_2) .$$

(2) Let C be the nodal curve over k obtained from $\mathbf{P}^1_k$ by identifying 0 and 1. Then

$$\pi_1^{\mathrm{cond,qs}}(C, c) \simeq \mathbf{Z} .$$

For more computations involving the van Kampen formula (but for Noohi groups), see [52].

7.37 Corollary (Künneth formula for the quasiseparated fundamental groups)**.** *Let k be a separably closed field and let X and Y be k-schemes such that X, Y, and $X \times_k Y$ satisfy the hypotheses of Notation 7.34. Let $\bar{z} \to X \times_k Y$ be a geometric point lying over geometric points $\bar{x} \to X$ and $\bar{y} \to Y$. If Y is proper or* $\mathrm{char}(k) = 0$*, then the natural homomorphism of condensed groups*

$$\pi_1^{\mathrm{cond,qs}}(X \times_k Y, \bar{z}) \to \pi_1^{\mathrm{cond,qs}}(X, \bar{x}) \times \pi_1^{\mathrm{cond,qs}}(Y, \bar{y})$$

is an isomorphism.

To prove this result, one can combine the van Kampen formula for $\pi_1^{\mathrm{cond,qs}}$ and the classical Künneth formula for $\pi_1^{\text{ét}}$ as in the proof of [52, Proposition 3.29], but this would require one to argue using the explicit relations appearing in the van Kampen theorem. To avoid it, it is beneficial to first apply the classical van Kampen in the groupoid form and only compute the π_1's at the very end. This is how we structure the proof below.

Proof. Fix integral hypercovers $\nu_{X,\bullet}, \nu_{Y,\bullet}$ by normal schemes of X and Y. Their product is again an integral hypercover of $X \times_k Y$ by normal schemes. Apply $\widehat{\Pi}_\infty^{\text{ét}}(-)$ to these diagrams and pass to colimits in $\mathrm{Cond}(\mathbf{Ani})$. The fixed geometric point $\bar{z}$ points them. Then 1-truncate and apply $\pi_1^{\mathrm{cond,qs}}(-)$ to both sides. We get a homomorphism of condensed groups

$$\pi_1(\operatorname*{colim}_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} \widehat{\Pi}_1^{\text{ét}}(X_m \times Y_m), *)^{\mathrm{qs}} \to \pi_1(\operatorname*{colim}_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} \widehat{\Pi}_1^{\text{ét}}(X_m) \times \widehat{\Pi}_1^{\text{ét}}(Y_m), *)^{\mathrm{qs}}$$

Using [41, Proposition A.1], we can replace $\operatorname{colim}_{[m]\in\mathbf{\Delta}^{\mathrm{op}}}$ by $\operatorname{colim}_{[m]\in\mathbf{\Delta}^{\mathrm{op}}_{\leqslant 2}}$. Apply the usual Künneth formula for $\pi_1^{\text{ét}}$ (c.f. [SGA 1, Exposé X, Corollaire 1.7 & Exposé XII, Proposition 4.6] or [35, §4]), which implies that $\widehat{\Pi}_1^{\text{ét}}(X_m \times Y_m) = \widehat{\Pi}_1^{\text{ét}}(X_m) \times \widehat{\Pi}_1^{\text{ét}}(Y_m)$, to get an isomorphism

$$\pi_1(\operatorname*{colim}_{[m]\in\mathbf{\Delta}^{\mathrm{op}}_{\leqslant 2}} \widehat{\Pi}_1^{\text{ét}}(X_m \times Y_m), *)^{\mathrm{qs}} \xrightarrow{\sim} \pi_1(\operatorname*{colim}_{[m]\in\mathbf{\Delta}^{\mathrm{op}}_{\leqslant 2}} \widehat{\Pi}_1^{\text{ét}}(X_m), *)^{\mathrm{qs}} \times \pi_1(\operatorname*{colim}_{[m]\in\mathbf{\Delta}^{\mathrm{op}}_{\leqslant 2}} \widehat{\Pi}_1^{\text{ét}}(Y_m), *)^{\mathrm{qs}} .$$

Now, using the equality $\pi_1^{\mathrm{cond,qs}} = \pi_1^{\text{ét}}$ on normal schemes and arguing via the van Kampen formula as in Theorem 7.35 to replace the fundamental groupoids by groups, we get that, e.g.,

$$\pi_1(\operatorname*{colim}_{[m]\in\mathbf{\Delta}^{\mathrm{op}}_{\leqslant 2}} \widehat{\Pi}_1^{\text{ét}}(X_m), *)^{\mathrm{qs}} = \pi_1^{\mathrm{cond,qs}}(X, \bar{x})$$

and similarly for Y and $X \times Y$. Note that $X^{2\nu}, X^{3\nu}$ (and similarly for $Y^{\cdots}$) might not be normal, but in the van Kampen formula all maps from $\pi_1^{\mathrm{cond,qs}}$ of (connected components) of those schemes will always factor though a profinite group (by normality of X^ν, Y^ν and $X^\nu \times Y^\nu$), so we were allowed to replace Π_1^{cond} by $\widehat{\Pi}_1^{\text{ét}}$ even for those non-normal schemes in the above computation (cf. similar argument appears in the proof of Theorem 7.35). This completes the proof. ☐

7.38 Corollary. *Let $K \supset k$ be an extension of separably closed fields, and let X be a k-scheme satisfying the hypotheses of Notation 7.34. If $\operatorname{char}(k) = 0$ or X is proper, then the projection $X_K \to X$ induces an isomorphism*

$$\pi_1^{\mathrm{cond,qs}}(X_K) \xrightarrow{\sim} \pi_1^{\mathrm{cond,qs}}(X)\,.$$

In the parlance of [49], the property of schemes established in Corollary 7.38 could be called "$\pi_1^{\mathrm{cond,qs}}$-properness".

7.39 Remark. In the context of anabelian geometry, it is sometimes beneficial to have a version of the Kurosh subgroup theorem available in the category of groups where our fundamental groups live, or at least its corollary: the characterization of maximal finite/compact/…subgroups of a free product as a "vertex subgroup" (i.e., one of the free summands up to conjugation). See, e.g., [61]. Proving such a result for the proétale fundamental group seems rather tricky due to the presence of Noohi completions. For $\pi_1^{\mathrm{cond,qs}}$, however, this can be done: see Proposition 7.40.

7.40 Proposition. *Let X be a scheme and $\bar{x}$ a geometric point such that*

$$\pi_1^{\mathrm{cond,qs}}(X,\bar{x}) \;\simeq *_i^{\mathrm{top}}\; G_i *^{\mathrm{top}} \mathbf{Z}^{*r}$$

where the G_i are profinite and $r \in \mathbf{N}$. Let H be a compact topological group and $\varphi : H \to \pi_1^{\mathrm{cond,qs}}(X,\bar{x})$ a continuous homomorphism. Then $\operatorname{im}(\varphi) \subset gG_ig^{-1}$ for some i and $g \in \pi_1^{\mathrm{cond,qs}}(X,\bar{x})$.

Proof. This follows follows from [63, Theorem 1]. ☐

7.41 Remark. We expect the assumptions of Proposition 7.40 to be satisfied, e.g., when X is a (semistable) curve over a separably closed field k, with $G_i = \pi_1^{\text{ét}}(X_i^\nu, \bar{x}_i)$, where $X = \sqcup_i X_i^\nu$ is the the normalization of X.

For $\pi_1^{\text{ét}}$ (or even $\pi_1^{\text{proét}}$), this is a classical computation using the van Kampen theorem when X is semistable (c.f. [73, Example 5.5] in the case of $\pi_1^{\text{ét}}$ or [54, Theorem 1.17] for $\pi_1^{\text{proét}}$) but with some care can be done for arbitrary curves, see [53, Theorem 2.27]. A similar computation (using Theorem 7.35) should extend this to $\pi_1^{\mathrm{cond,qs}}$.

8 The Noohi completion of the condensed fundamental group

The goal of this section is to recover the proétale fundamental group $\pi_1^{\text{proét}}(X,\bar{x})$ of [10, §7] of a topologically noetherian scheme X from the condensed fundamental group $\pi_1^{\mathrm{cond}}(X,\bar{x})$. The main input needed for this is the observation that all weakly locally constant sheaves in the sense of [10, Definition 7.3.1] can be recovered from $\pi_1^{\mathrm{cond}}(X,\bar{x})$. We prove a stronger derived version of that result in §8.1. In §8.2, we explain how to Noohi complete condensed groups and show that the Noohi completion of $\pi_1^{\mathrm{cond}}(X,\bar{x})$ is indeed the proétale fundamental group of [10, §7].

8.1 Recovering weakly locally constant sheaves

In this subsection, we explain how to recover weakly locally constant proétale sheaves on a scheme X as representations of the condensed homotopy type. The following is a generalization of [10, Definition 7.3.1] to sheaves of anima:

8.1 Recollection. Recall that for a qcqs scheme X there is a canonical algebraic morphism $\mathrm{Sh}(\pi_0(X)) \to X_{\acute{e}t}$ induced by sending a clopen in $\pi_0(X)$ to its preimage in X. Furthermore $F \in X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}}$ is said to be *locally weakly constant* if there is a proétale cover $\{U_i \to X\}_{i \in I}$ by qcqs schemes such that each $F|_{U_i}$ is in the image of the canonical algebraic morphism

$$\mathrm{Sh}(\pi_0(U_i)) \longrightarrow U^{\mathrm{hyp}}_{i,\acute{e}t} \xrightarrow{\nu^*} U^{\mathrm{hyp}}_{i,\mathrm{pro\acute{e}t}}\,.$$

We denote the full subcategory of $X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}}$ spanned by the locally weakly constant sheaves by $\mathrm{wLoc}(X)$.

8.2 Definition. We define the condensed ∞-category $\mathbf{Ani}^{\mathrm{ult}}$ by the assignment

$$S \mapsto \mathrm{Sh}(S)$$

for every profinite set S. Similarly, we refer to the 0-truncated version of this condensed ∞-category by $\mathbf{Set}^{\mathrm{ult}}$.

For every profinite set S, there is a canonical fully faithful comparison functor

$$c_S^* \colon \mathrm{Sh}(S) \hookrightarrow \mathrm{Cond}(\mathbf{Ani})_{/S}$$

which is part of a geometric morphism of ∞-topoi [32, Sections 3.2]. As the comparison map is natural in S, see [32, Lemma 3.16], it induces an embedding

$$\mathbf{Ani}^{\mathrm{ult}} \hookrightarrow \mathbf{Cond}(\mathbf{Ani})$$

of condensed ∞-categories.

8.3 Remark. The superscript 'ult' comes from the word *ultrastructure*. Any category with filtered colimits and infinite products can be canonically upgraded to an ultracategory by equipping it with the *categorical ultrastructure*, see [55, Example 1.3.8]. In [55, Construction 4.1.1] Lurie explains how to regard ultra categories as condensed categories. Furthermore it follows from [55, Theorem 3.4.4] that the image of **Set** equipped with the categorical ultrastructure is precisely $\mathbf{Set}^{\mathrm{ult}}$.

8.4 Recollection. Using [80, Corollary 1.2], we have a fully faithful functor

$$b^* \colon \mathrm{Fun}^{\mathrm{cts}}(\Pi_\infty^{\mathrm{cond}}(X), \mathbf{Cond}(\mathbf{Ani})) \longrightarrow \mathrm{Fun}^{\mathrm{cts}}\big(\mathrm{Gal}(X), \mathbf{Cond}(\mathbf{Ani})\big) \simeq X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}}$$

given by precomposition with the localization $b \colon \mathrm{Gal}(X) \to \mathrm{B}^{\mathrm{cond}}\mathrm{Gal}(X) = \Pi_\infty^{\mathrm{cond}}(X)$ as in the proof of Proposition 3.36.

8.5 Theorem. *Let X be a qcqs scheme. The composite fully faithful functor*

$$\mathrm{Fun}^{\mathrm{cts}}\big(\Pi_\infty^{\mathrm{cond}}(X), \mathbf{Ani}^{\mathrm{ult}}\big) \hookrightarrow \mathrm{Fun}^{\mathrm{cts}}\big(\Pi_\infty^{\mathrm{cond}}(X), \mathbf{Cond}(\mathbf{Ani})\big) \xhookrightarrow{b^*} X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}} \tag{8.6}$$

has image the full subcategory $\mathrm{wLoc}(X)$ *of locally weakly constant sheaves.*

The idea of the proof of Theorem 8.5 is to show it first in the case of w-contractible affine schemes and then conclude by proétale hyperdescent.

8.7 Lemma. *Let W be a w-contractible affine scheme. Then the fully faithful functor*

$$\mathrm{Fun}^{\mathrm{cts}}(\pi_0(W), \mathbf{Ani}^{\mathrm{ult}}) \to W^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}}$$

has image $\mathrm{wLoc}(W)$.

Proof. Recall that since W is w-contractible, $\Pi^{\mathrm{cond}}_{\infty}(W) \simeq \pi_0(W)$. Moreover, since $\pi_0(W)$ is a profinite set, the Yoneda lemma implies that

$$\mathrm{Fun}^{\mathrm{cts}}(\pi_0(W), \mathbf{Ani}^{\mathrm{ult}}) \simeq \mathbf{Ani}^{\mathrm{ult}}(\pi_0(W)) \simeq \mathrm{Sh}(\pi_0(W))$$

and the given functor is identified with the pullback functor

$$\mathrm{Sh}(\pi_0(W)) \hookrightarrow W^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}}$$

along $W \to \pi_0(W)$. Therefore it lands in $\mathrm{wLoc}(W)$ by definition and it remains to show surjectivity. Let $F \in \mathrm{wLoc}(W)$. Then there is a proétale cover $p\colon\ U \to W$ such that p^*F is in the image of $\mathrm{Sh}(\pi_0(U)) \to U^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}}$. Since W is w-contractible, we can pick a section $s\colon\ W \to U$ and since the diagram

$$\begin{tikzcd} W \arrow[r, "\nu"] \arrow[d, "s"'] & \pi_0(W) \arrow[d, "\pi_0(s)"] \\ U \arrow[r] & \pi_0(U) \end{tikzcd}$$

commutes, we see that $F = s^*p^*F$ is in the image of ν^*. ☐

Proof of Theorem 8.5. As we have a chain of fully faithful functors (8.6), we regard

$$\mathrm{Fun}^{\mathrm{cts}}(\Pi^{\mathrm{cond}}_{\infty}(X), \mathbf{Ani}^{\mathrm{ult}})$$

as a full subcategory of $X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}}$. It remains to show that this full subcategory agrees with the full subcategory $\mathrm{wLoc}(X)$. Since the assignment $Y \mapsto \Pi^{\mathrm{cond}}_{\infty}(Y)$ is a hypercomplete proétale cosheaf, the assignment

$$Y \mapsto \mathrm{Fun}(\Pi^{\mathrm{cond}}_{\infty}(Y), \mathbf{Ani}^{\mathrm{ult}})$$

is in a fact a subsheaf of the proétale hypersheaf $Y \mapsto Y^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}}$. Furthermore, by definition, the assignment

$$Y \mapsto \mathrm{wLoc}(Y)$$

is subsheaf of the proétale hypersheaf $Y \mapsto Y^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}}$. Therefore, it suffices to see that they agree on w-contractibles, which is the content of Lemma 8.7. ☐

8.2 Recovering the proétale fundamental group

To define Noohi completion for condensed groups, we will use the following left adjoint.

8.8 Recollection. The canonical functor Grp(**Top**) → Cond(**Grp**) admits a left adjoint

$$(-)^{\mathrm{top}} \colon \ \mathrm{Cond}(\mathbf{Grp}) \to \mathrm{Grp}(\mathbf{Top}) \, .$$

Note, however, that in general it is not the restriction of the left adjoint "underlying topological space" functor

$$(-)(*)_{\mathrm{top}} \colon \ \mathrm{Cond}(\mathbf{Set}) \to \mathbf{Top}$$

to condensed groups, as this latter functor does not preserve products.

It turns out that some insight into what $(-)(*)_{\mathrm{top}}$ does can be gained in terms of *quasitopological groups*.

8.9 Remark. Recall that a *quasitopological group* is a topological space G with an abstract group structure such that:

(1) The inversion operation $G \to G$ given by $g \mapsto g^{-1}$ is continuous.

(2) For each $h \in G$, the translations $l_h, r_h \colon G \to G$, given by $g \mapsto gh$ and $g \mapsto hg$, are continuous.

The embedding Grp(**Top**) ⊂ **qTopGrp** of topological groups into quasitopological groups admits a left adjoint

$$\tau \colon \ \mathbf{qTopGrp} \to \mathrm{Grp}(\mathbf{Top})$$

that moreover preserves the underlying abstract group and only affects the topology, cf. [12, Lemma 3.2 & Theorem 3.8].

While the functor $(-)(*)_{\mathrm{top}}$ does not provide (after restriction) an adjoint between Cond(**Grp**) and Grp(**Top**), its image still lands in **qTopGrp**. This is essentially because the condition of continuity of the inversion and translation maps does not involve forming a product. Thus, after restriction, we can consider

$$(-)(*)_{\mathrm{top}} \colon \ \mathrm{Cond}(\mathbf{Grp}) \to \mathbf{qTopGrp}$$

Postcomposing with τ, we get a functor

$$\tau \circ (-)(*)_{\mathrm{top}} \colon \ \mathrm{Cond}(\mathbf{Grp}) \to \mathrm{Grp}(\mathbf{Top})$$

One can then quite directly verify (see [56, Proposition 1.3.16] for details) the following result:

The composition $\tau \circ (-)(*)_{\mathrm{top}}$ is a left adjoint to the "associated condensed group" functor. Visually

$$\tau \circ (-)(*)_{\mathrm{top}} \colon \ \mathrm{Cond}(\mathbf{Grp}) \rightleftarrows \mathrm{Grp}(\mathbf{Top}) \colon \underline{(-)}$$

Let us denote this composed functor by $(-)^{\mathrm{top}}$. It follows from this discussion that for $G \in$ Cond(**Grp**), the abstract group $G(*)$ and the underlying group of G^{top} match.

8.10 Recollection [10, §7.1]**.** For a topological group G, let $F_G \colon G$-**Set** → **Set** denote the forgetful functor from sets with a continuous action by G to abstract sets. We say G is *Noohi* if the canonical continuous map

$$G \to \mathrm{Aut}(F_G)$$

is a homeomorphism of groups. Here, $\mathrm{Aut}(F_G)$ is topologized using the compact-open topology on groups $\mathrm{Aut}(F_G(M))$ for $M \in G$-**Set**.

This type of a topological group is useful when one wants to generalize Galois theory of Grothendieck to allow infinite fibers (cf. the “infinite Galois theory” of [10, §7.2]). This formalism was used to define the proétale fundamental group $\pi_1^{\text{proét}}$ of a scheme in §7.4 of *loc. cit.*. The group $\pi_1^{\text{proét}}$ is Noohi. Similarly, the fundamental group of de Jong in rigid geometry (see [47]) and its later generalizations (see [2] and [1]) are all Noohi.

Noohi groups can also be characterized in purely topological terms as Hausdorff, Raĭkov complete groups such that open subgroups form a fundamental system of neighborhoods at 1.

The inclusion $\mathbf{Grp}^{\text{Noohi}} \subset \operatorname{Grp}(\mathbf{Top})$ admits a left adjoint $(-)^{\text{Noohi}}$, called “Noohi completion”, given by

$$\operatorname{Grp}(\mathbf{Top}) \ni G \mapsto \operatorname{Aut}(F_G) \in \mathbf{Grp}^{\text{Noohi}} .$$

See [52, §2] for this and some other properties of Noohi groups and completions.

We now want to extend Noohi completion to condensed groups.

8.11 Definition. Let $G \in \operatorname{Cond}(\mathbf{Grp})$. We define

$$G^{\text{Noohi}} = \left(G^{\text{top}}\right)^{\text{Noohi}} \in \mathbf{Grp}^{\text{Noohi}}$$

to be the Noohi completion of G.

8.12 Theorem. *Let X be a topologically noetherian scheme and $\bar{x} \to X$ a geometric point. Then there is a natural isomorphism*

$$(\pi_1^{\text{cond}}(X,\bar{x})^{\text{top}})^{\text{Noohi}} \cong \pi_1^{\text{proét}}(X,\bar{x}) .$$

8.13 Remark. For a condensed group G, one can also define a version of Noohi completion $G^{\text{Noohi}} \in \operatorname{Cond}(\mathbf{Grp})$ directly as a condensed group without passing through $(-)^{\text{top}}$. More precisely one can define $G^{\text{Noohi}}$by the assignment

$$S \mapsto \operatorname{Aut}\left(\operatorname{Fun}^{\text{cts}}(\mathrm{B}G, \mathbf{Set}^{\text{ult}}) \to \mathbf{Set} \xrightarrow{\Gamma_S^*} \operatorname{Sh}(S)\right) .$$

It turns out that the two definitions match, that is, one can check

$$G^{\text{Noohi}} \cong \underline{(G^{\text{top}})^{\text{Noohi}}} .$$

We will not need this observation in this article.

The main input that we need is the following:

8.14 Lemma. *Let G be a condensed group with condensed classifying anima* $\mathrm{B}G$. *There is a natural equivalence of categories*

$$\operatorname{Fun}^{\text{cts}}(\mathrm{B}G, \mathbf{Set}^{\text{ult}}) \to G^{\text{top}}\text{-}\mathbf{Set}$$

that is compatible with the forgetful functors to **Set**.

Proof. We first prove the following: the category $\operatorname{Fun}^{\text{cts}}(\mathrm{B}G, \mathbf{Set}^{\text{ult}})$ is equivalent to the category of pairs (M, α) where $M \in \mathbf{Set}$ and $\alpha : G \to \underline{\operatorname{Aut}(M)^{\text{top}}}$ is a map of condensed groups. Here, the topological group $\operatorname{Aut}(M)^{\text{top}}$ is the group of automorphisms $\operatorname{Aut}(M)$ topologized via the compact-open topology. A map $(M,\alpha) \to (N,\beta)$ is given by a map $f : M \to N$ such that the square

$$\begin{tikzcd}
G \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \underline{\mathrm{Aut}(M)^{\mathrm{top}}} \arrow[d, "f_*"] \\
\underline{\mathrm{Aut}(N)^{\mathrm{top}}} \arrow[r, "f^*"'] & \underline{\mathrm{Hom}_{\mathbf{Top}}(M,N)^{\mathrm{top}}}
\end{tikzcd}$$

commutes (here $\mathrm{Hom}_{\mathbf{Top}}(M,N)^{\mathrm{top}}$ is again topologized via the compact-open topology). If this description holds, the claim follows: by adjunction, a homomorphism $G \to \underline{\mathrm{Aut}(M)^{\mathrm{top}}}$ can be uniquely identified with a homomorphism $G^{\mathrm{top}} \to \mathrm{Aut}(M)^{\mathrm{top}}$ of topological groups and similarly for N. Despite the fact that $\underline{\mathrm{Hom}_{\mathbf{Top}}(M,N)^{\mathrm{top}}}$ is not a group, Remark 8.9 shows that the diagram

$$\begin{tikzcd}
G^{\mathrm{top}} \arrow[r, "\alpha^{\mathrm{top}}"] \arrow[d, "\beta^{\mathrm{top}}"'] & \mathrm{Aut}(M)^{\mathrm{top}} \arrow[d, "f_*"] \\
\mathrm{Aut}(N)^{\mathrm{top}} \arrow[r, "f^*"'] & \mathrm{Hom}_{\mathbf{Top}}(M,N)^{\mathrm{top}}
\end{tikzcd}$$

nevertheless commutes. This shows the desired equivalence with the category of G^{top}-**Set**.

The fully faithful functor $\mathrm{Fun}^{\mathrm{cts}}(\mathrm{B}G, \mathbf{Set}^{\mathrm{ult}}) \hookrightarrow \mathrm{Fun}^{\mathrm{cts}}(\mathrm{B}G, \mathrm{\mathbf{Cond}}(\mathbf{Set}))$ fits into a cartesian square

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{cts}}(\mathrm{B}G, \mathbf{Set}^{\mathrm{ult}}) \arrow[r, "\mathrm{ev}_*"] \arrow[d, hook] & \mathbf{Set} \arrow[d, hook] \\
\mathrm{Fun}^{\mathrm{cts}}(\mathrm{B}G, \mathbf{Cond}(\mathbf{Set})) \arrow[r, "\mathrm{ev}_*"'] & \mathrm{Cond}(\mathbf{Set}),
\end{tikzcd}$$

where the horizontal arrows are given by pullback along $* \to \mathrm{B}G$. Indeed, this follows as the functors

$$\mathrm{Fun}^{\mathrm{cts}}(-, \mathbf{Set}^{\mathrm{ult}}), \mathrm{Fun}^{\mathrm{cts}}(-, \mathbf{Cond}(\mathbf{Set})) : \ \mathrm{Cond}(\mathbf{Ani})^{\mathrm{op}} \to \mathbf{Cat}$$

are sheaves and $* \to \mathrm{B}G$ is a cover in $\mathrm{Cond}(\mathbf{Ani})$. Furthermore, applying $\mathrm{Fun}^{\mathrm{cts}}(-, \mathbf{Cond}(\mathbf{Set}))$ to the Čech-nerve of $* \to \mathrm{B}G$, we get an equivalence

$$\mathrm{Fun}^{\mathrm{cts}}(\mathrm{B}G, \mathbf{Cond}(\mathbf{Set})) \simeq \lim \Big(\mathrm{Cond}(\mathbf{Set}) \rightrightarrows \mathrm{Cond}(\mathbf{Set})_{/G} \substack{\rightarrow\\[-3pt]\rightarrow\\[-3pt]\rightarrow} \mathrm{Cond}(\mathbf{Set})_{/G} \Big)$$

using that by [80, Corollary 3.20]), for a condensed set A, there is a natural equivalence of categories

$$\mathrm{Fun}^{\mathrm{cts}}(A, \mathbf{Cond}(\mathbf{Set})) \simeq \mathrm{Cond}(\mathbf{Set})_{/A}.$$

Explicitly unwinding the descent datum, we see that $\mathrm{Fun}^{\mathrm{cts}}(\mathrm{B}G, \mathbf{Cond}(\mathbf{Set}))$ is equivalent to the usual category of condensed sets with an action by the condensed group G. In other words, its objects are condensed sets A together with a map $G \to \underline{\mathrm{Aut}}(A)$ of condensed groups and the maps are defined as above. Here $\underline{\mathrm{Aut}}(A)$ is the maximal condensed subgroup of the condensed monoid $\underline{\mathrm{Hom}}(A,A)$ given by the internal hom in $\mathrm{Cond}(\mathbf{Set})$. Thus, the proof will be complete if for a set M, we can show that there is a canonical isomorphism

$$\underline{\mathrm{Aut}}(M,M) \xrightarrow{\sim} \underline{\mathrm{Aut}(M,M)^{\mathrm{top}}} .$$

But for this, we observe that we have a canonical isomorphism $\underline{\mathrm{Hom}}(M,M) \cong \underline{\mathrm{Hom}_{\mathbf{Top}}(M,M)}$ under which the corresponding condensed subgroups of automorphisms agree. This completes the proof. ☐

Proof of Theorem 8.12. We may assume that X, and therefore $\Pi_\infty^{\mathrm{cond}}(X)$, is connected by Corollary 4.19 as X has finitely many irreducible components. It follows from Theorem 8.5 that we have a chain of natural equivalences

$$\begin{aligned}\mathrm{Fun}^{\mathrm{cts}}(\mathrm{B}\pi_1^{\mathrm{cond}}(X,\bar{x}), \mathbf{Set}^{\mathrm{ult}}) &\simeq \mathrm{Fun}^{\mathrm{cts}}(\Pi_1^{\mathrm{cond}}(X), \mathbf{Set}^{\mathrm{ult}})\\ &\simeq \mathrm{Fun}^{\mathrm{cts}}(\Pi_\infty^{\mathrm{cond}}(X), \mathbf{Set}^{\mathrm{ult}})\\ &\simeq \mathrm{wLoc}(X)_{\leq 0}\\ &\simeq \pi_1^{\mathrm{pro\acute{e}t}}(X,\bar{x})\text{-}\mathbf{Set}\end{aligned}$$

that is compatible with the forgetful functors to **Set**. Here, the last equivalence is due to the definition of $\pi_1^{\mathrm{pro\acute{e}t}}(X,\bar{x})$ in [10, Definition 7.4.2] coupled with Lemmas 7.3.9 and 7.4.1 in *loc. cit.*. Thus, Lemma 8.14 shows that there is a natural equivalence

$$\pi_1^{\mathrm{cond}}(X,\bar{x})^{\mathrm{top}}\text{-}\mathbf{Set} \simeq \pi_1^{\mathrm{pro\acute{e}t}}(X,\bar{x})\text{-}\mathbf{Set}\,.$$

In particular, both groups have the same Noohi completion. Since $\pi_1^{\mathrm{pro\acute{e}t}}(X,\bar{x})$ is Noohi complete by [10, Theorem 7.2.5], the claim follows. ☐

A Rings of continuous functions & Čech–Stone compactification

by Bogdan Zavyalov

The main goal of this appendix is to give a self-contained account for the identification of the Čech–Stone compactification of a topological space X with the maximal spectrum of the ring of continuous functions on X.

This identification has already been established in [18] using the notion of pm-ring. In this appendix, we follow the ideas already present in [18]. We do not claim originality of any results in this appendix. Instead, we hope that this appendix gives a self-contained and reader-friendly exposition of some ideas from [18] and [27].

Throughout this appendix, we denote by $\mathbf{R}$ (resp. $\mathbf{C}$) the topological ring of real numbers (resp. complex numbers) with the Euclidean topology. For a topological space X, we denote by $\mathrm{C}(X, \mathbf{R})$ (resp. $\mathrm{C}(X, \mathbf{C})$) the ring of real-valued (resp. complex-valued) continuous functions on X.

Many of the results in this appendix also appear in [78; 79; 77].

A.1 Main constructions

The main goal of this subsection is to introduce some constructions that will be used in the rest of this appendix. We also study their basic properties.

A.1 Construction. Let X be a topological space.

(1) For each point $x \in X$, we define the *evaluation functional* $\mathrm{ev}_x \colon \mathrm{C}(X, \mathbf{R}) \to \mathbf{R}$ by the formula

$$\mathrm{ev}_x(f) \coloneqq f(x) .$$

(2) We define the map

$$\iota_X \colon X \to \mathrm{Spec}\big(\mathrm{C}(X, \mathbf{R})\big)$$

to be the unique map that sends each point $x \in X$ to $\ker(\mathrm{ev}_x)$.

A.2 Remark. The map ι_X is clearly functorial in X.

For our later convenience, we record some basic properties of ι_X.

A.3 Lemma. *Let X be a topological space.*

(1) *The natural map $\iota_X \colon X \to \mathrm{Spec}\big(\mathrm{C}(X, \mathbf{R})\big)$ is continuous;*

(2) *the image of $\iota_X(X) \subset \mathrm{Spec}\big(\mathrm{C}(X, \mathbf{R})\big)$ is a dense subset;*

(3) *the map ι_X factors through $\mathrm{MSpec}\big(\mathrm{C}(X, \mathbf{R})\big)$.*

Proof. In order to see the first claim, it suffices to show that $\iota_X^{-1}\big(\mathrm{D}(f)\big)$ is an open subset of X for every $f \in \mathrm{C}(X, \mathbf{R})$. This follows immediately from the formula $\iota_X^{-1}\big(\mathrm{D}(f)\big) = \{x \in X \mid f(x) \neq 0\}$ and the assumption that f is continuous.

Now we prove the second claim. Let $Z \coloneqq \mathrm{V}(I) \subset \mathrm{Spec}\big(\mathrm{C}(X, \mathbf{R})\big)$ be a closed subset containing $\iota_X(X)$. Then the construction of ι_X implies that, for every $f \in I$, we have $0 = \mathrm{ev}_x(f) = f(x)$ for all $x \in X$. Thus $f = 0$ and so we conclude that $Z = \mathrm{V}(0) = \mathrm{Spec}\big(\mathrm{C}(X, \mathbf{R})\big)$.

To justify the last claim, it is enough to prove that $\ker(\mathrm{ev}_x)$ is a maximal ideal for every $x \in X$. For this, it suffices to show that ev_x is surjective. Fix a constant $c \in \mathbf{R}$ and denote by $\underline{c}$ the corresponding constant function on X. Then the surjectivity of ev_x follows immediately from the observation that $\mathrm{ev}_x(\underline{c}) = c$. ☐

A.4 Remark. In what follows, we slightly abuse the notation and also denote by ι_X the natural morphism $\iota_X : X \to \mathrm{MSpec}\big(\mathrm{C}(X, \mathbf{R})\big)$.

We will show later in this appendix that ι_X is a homeomorphism when X is a compact Hausdorff topological space.

A.5 Warning. The map ι_X is neither injective nor surjective for a general topological space X.

A.2 pm-rings

In this subsection, we introduce the notion of *pm-rings* following [18]. Then we show that the natural inclusion $\mathrm{MSpec}(A) \hookrightarrow \mathrm{Spec}(A)$ admits a continuous retract for a pm-ring A. As a consequence, we deduce that $\mathrm{MSpec}(A)$ is a compact Hausdorff space for any pm-ring A. We will use the results of this section to relate the Čech–Stone compactification of an arbitrary topological space X to the maximal spectrum of the ring of continuous functions on X.

A.6 Definition [18]**.** A ring A is a *pm-ring* if every prime ideal $\mathfrak{p} \subset A$ is contained in a unique maximal ideal $\mathfrak{p} \subset \mathfrak{m}_{\mathfrak{p}} \subset A$.

A.7 Definition. For a pm-ring A, we define the *retract map* $r_A : \mathrm{Spec}(A) \to \mathrm{MSpec}(A)$ as the unique map that sends a point x to its unique closed specialization (equivalently, it sends each prime ideal $\mathfrak{p}$ to the unique maximal ideal $\mathfrak{m}_{\mathfrak{p}}$ containing $\mathfrak{p}$). When there is no possibility of confusion, we will denote the map r_A simply by r.

A.8 Remark. Below, we present a proof that r_A is always continuous for a pm-ring A. This beautiful proof is due to De Marco and Orsatti. However, we want to emphasize that, a priori, it is absolutely not clear whether the map r_A has to be continuous or not. In fact, the author finds it quite surprising and is not aware of any one-line proof of this fact.

A.9 Theorem [18, Theorem 1.2]**.** *Let A be a pm-ring. Then $r : \mathrm{Spec}(A) \to \mathrm{MSpec}(A)$ is a continuous retract of the natural embedding $\iota : \mathrm{MSpec}(A) \hookrightarrow \mathrm{Spec}(A)$.*

In fact, [18, Theorem 1.2] shows that A is a pm-ring if and only if ι admits a continuous retract (and r is the unique continuous retract in this case). However, since we never need the other direction and it is significantly easier, we decided not to include it in this exposition.

Proof. Throughout this proof, we denote by $\mathrm{V}_{\mathrm{Spec}}(I) \subset \mathrm{Spec}(A)$ the vanishing locus of an ideal I inside $\mathrm{Spec}(A)$, and by $\mathrm{V}_{\mathrm{Max}}(I) := \mathrm{V}_{\mathrm{Spec}}(I) \cap \mathrm{MSpec}(A)$ the vanishing locus of I inside $\mathrm{MSpec}(A)$.

By construction, we know that $r \circ \iota = \mathrm{id}$. So the only thing we really need to show is that the map r is continuous. We fix a closed subset $Z \subset \mathrm{MSpec}(A)$ and define

$$I := \bigcap_{\mathfrak{m} \in Z} \mathfrak{m} \qquad \text{and} \qquad J := \bigcap_{\substack{\mathfrak{p} \subset A \\ r(\mathfrak{p}) \in Z}} \mathfrak{p}\,.$$

For the purpose of proving continuity of r, it is enough to show that $r^{-1}(Z) = \mathrm{V}_{\mathrm{Spec}}(J)$. Clearly, $r^{-1}(Z) \subset \mathrm{V}_{\mathrm{Spec}}(J)$. Therefore, after unravelling all the definitions, we see that it suffices to show that, for any prime ideal $\mathfrak{p} \subset A$ such that $J \subset \mathfrak{p}$, we have $r(\mathfrak{p}) \in Z$.

Step 1: We show $Z = \mathrm{V}_{\mathrm{Max}}(I)$. Since Z is closed, we know that $Z = \mathrm{V}_{\mathrm{Max}}(K)$ for some ideal $K \subset A$. By construction, for any $\mathfrak{m} \in Z$, we have $K \subset \mathfrak{m}$. In particular, $K \subset I = \bigcap_{\mathfrak{m} \in Z} \mathfrak{m}$. Thus, $\mathrm{V}_{\mathrm{Max}}(I) \subset \mathrm{V}_{\mathrm{Max}}(K) = Z$. On the other hand, the definition of I implies that $Z \subset \mathrm{V}_{\mathrm{Max}}(I)$. Therefore, we conclude that

$$\mathrm{V}_{\mathrm{Max}}(I) \subset \mathrm{V}_{\mathrm{Max}}(K) = Z \subset \mathrm{V}_{\mathrm{Max}}(I)\,.$$

This implies that $\mathrm{V}_{\mathrm{Max}}(I) = Z$.

Now we set $M \coloneqq \bigcup_{\mathfrak{m} \in Z} \mathfrak{m}$. We note that $1 \notin M$, so $M \neq A$. We warn the reader that the set M is not generally an ideal in A.

Step 2: Let $\mathfrak{p} \subset M$ be a prime ideal in A. Then $r(\mathfrak{p}) \in Z$. Since $\mathfrak{p} \subset M$ and $I = \bigcap_{\mathfrak{m} \in Z} \mathfrak{m}$, we conclude that $\mathfrak{p} + I \subset M \neq A$. Thus, we can find a maximal ideal $\mathfrak{n} \subset A$ such that

$$\mathfrak{p} \subset \mathfrak{p} + I \subset \mathfrak{n}\,.$$

Therefore, $r(\mathfrak{p}) = \mathfrak{n}$. Since $I \subset \mathfrak{n}$, Step 1 ensures that $\mathfrak{n} \in Z$. This shows that $r(\mathfrak{p}) \in Z$.

Step 3: Let $J \subset \mathfrak{p}$ be a prime ideal in A. Then $r(\mathfrak{p}) \in Z$. Since each prime ideal is contained in a unique maximal ideal, it suffices to find a prime ideal $\mathfrak{q} \subset \mathfrak{p}$ such that $\mathfrak{q} \subset M$; then Step 2 implies that $r(\mathfrak{p}) = r(\mathfrak{q}) \in Z$.

Now we choose any $t \in A \smallsetminus \mathfrak{p}$ and $s \in A \smallsetminus M$. Then $ts \neq 0$ since otherwise it would imply that

$$t \in \bigcap_{\mathfrak{m} \in Z} \mathfrak{m} = J \subset \mathfrak{p}\,.$$

Hence, the multiplicative system

$$S = \{\, ts \mid t \in A \smallsetminus \mathfrak{p} \text{ and } s \in A \smallsetminus M \,\}$$

does not contain 0. Therefore, the localization $A[S^{-1}]$ is nonzero. Thus, any maximal ideal in $A[S^{-1}]$ defines a prime ideal $\mathfrak{q} \subset A$ disjoint from S. Since $1 \in A \smallsetminus \mathfrak{p}$ and $1 \in A \smallsetminus M$, we conclude that $\mathfrak{q} \subset \mathfrak{p} \cap M$, finishing the proof. ☐

A.10 Corollary. *Let A be a pm-ring. Then* MSpec(A) *is a compact Hausdorff space.*

Proof. Theorem A.9 constructs a continuous surjective map $r \colon \mathrm{Spec}(A) \to \mathrm{MSpec}(A)$. Since Spec($A$) is quasicompact and images of quasicompact spaces are quasicompact, MSpec(A) is also quasicompact.

Now we show that MSpec(A) is Hausdorff. First, [STK, Tag 0904] implies that it suffices to show that, for any two closed points $x, y \in \mathrm{Spec}(A)$, there does not exist a point $z \in \mathrm{Spec}(A)$ which specializes to both x and y. This follows immediately from the fact that every point of Spec(A) specializes to a unique closed point. ☐

A.11 Definition. Let $f \colon A \to B$ be a homomorphism between pm-rings. We define the *associated morphism of maximal spectra* $\mathrm{MSpec}(f) \colon \mathrm{MSpec}(B) \to \mathrm{MSpec}(A)$ as the composition

$$\begin{tikzcd} \mathrm{MSpec}(B) \arrow[r, "\iota_B"] & \mathrm{Spec}(B) \arrow[r, "\mathrm{Spec}(f)"] & \mathrm{Spec}(A) \arrow[r, "r_A"] & \mathrm{MSpec}(A)\,. \end{tikzcd}$$

A.12 Warning. In general, for a ring homomorphism $A \to B$, the induced map of spectra $\mathrm{Spec}(f) \colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ does not send MSpec(B) to MSpec(A). This does not even hold for a general homomorphism of pm-rings. Indeed, consider a rank 2 valuation ring V with the fraction field K and a rank-1 localization $\mathcal{O}$. Then the map $\mathrm{Spec}(\mathcal{O}) \to \mathrm{Spec}(V)$, induced by the inclusion $V \subset \mathcal{O}$, sends the closed point of Spec($\mathcal{O}$) to a non-closed point of Spec(V).

A.3 Rings of continuous functions

The main goal of this section is to show that the rings of continuous functions $\mathrm{C}(X, \mathbf{R})$ and $\mathrm{C}(X, \mathbf{C})$ are pm-rings for any topological space X. This will be the crucial ingredient in showing that the Čech–Stone compactification βX is homeomorphic to $\mathrm{MSpec}\big(\mathrm{C}(X, \mathbf{R})\big)$.

We do not claim originality of any results of this subsection. In fact, our presentation that $\mathrm{C}(X, \mathbf{R})$ is a pm-ring follows [27, Theorem 2.11] quite closely. The case of $\mathrm{C}(X, \mathbf{C})$ seems to be missing in [27].

Throughout the section, we fix a topological space X.

A.13 Definition. Let $f \in \mathrm{C}(X, \mathbf{R})$ be a continuous function. Its *vanishing locus* is the set

$$\mathrm{V}_X(f) \coloneqq \{\, x \in X \mid f(x) = 0 \,\} .$$

A.14 Definition. For a subset $S \subset \mathrm{C}(X, \mathbf{R})$, the *collection of its zero sets* is the subset

$$\mathrm{V}_X[S] \coloneqq \{\mathrm{V}_X(f) \mid f \in S\} \subset \mathrm{Sub}(X)$$

of the set of all vanishing loci of elements in S.[7] For brevity, we put $\mathrm{V}_X[X] \coloneqq \mathrm{V}_X[\mathrm{C}(X, \mathbf{R})]$ the set of all vanishing loci of continuous functions on X.

A.15 Lemma [27, Theorem 2.3]**.** *Let $I \subset \mathrm{C}(X, \mathbf{R})$ be an ideal and let $Z_1, Z_2 \in \mathrm{V}_X[I]$. Then*

(1) *$Z_1 \cap Z_2 \in \mathrm{V}_X[I]$;*

(2) *if $Z \in \mathrm{V}_X[X]$ and $Z_1 \subset Z$, then $Z \in \mathrm{V}_X[I]$.*

Proof. Let $Z_1 = \mathrm{V}_X(f_1)$, $Z_2 = \mathrm{V}_X(f_1)$, and $Z = \mathrm{V}_X(f)$ for $f_1, f_2 \in I$ and $f \in \mathrm{C}(X, \mathbf{R})$. The first claim follows immediately from the observation that

$$Z_1 \cap Z_2 = \mathrm{V}_X(f_1) \cap \mathrm{V}_X(f_2) = \mathrm{V}_X(f_1^2 + f_2^2) \in \mathrm{V}_X[I].$$

The second claim follows immediately from the observation that

$$Z = Z_1 \cup Z = \mathrm{V}_X(f_1) \cup \mathrm{V}_X(f) = \mathrm{V}_X(f_1 f) \in \mathrm{V}_X[I]. \qquad \square$$

A.16 Definition. An ideal $I \subset \mathrm{C}(X, \mathbf{R})$ is a *zs-ideal* if $\mathrm{V}_X(f) \in \mathrm{V}_X[I]$ implies $f \in I$.

A.17 Remark. Usually, zs-ideals are called z-ideals. We prefer to avoid this name for obvious reasons.

A.18 Theorem [27, Theorem 2.5]**.** *Let $\mathfrak{m} \subset \mathrm{C}(X, \mathbf{R})$ be a maximal ideal. Then $\mathfrak{m}$ is a zs-ideal.*

Proof. We denote by $I_{\mathfrak{m}} \subset \mathrm{C}(X, \mathbf{R})$ the subset of continuous functions whose vanishing locus is equal to a vanishing locus of a function in $\mathfrak{m}$, i.e.,

$$I_{\mathfrak{m}} = \{f \in \mathrm{C}(X, \mathbf{R}) \mid \mathrm{V}_X(f) \in \mathrm{V}_X[\mathfrak{m}]\} . \tag{A.19}$$

We first show that $\mathcal{I}(\mathrm{V}_X[\mathfrak{m}])$ is an ideal.

Now Lemma A.15 implies that $I_{\mathfrak{m}}$ is an ideal. We pick continuous functions $f, g \in I_{\mathfrak{m}}$ and $h \in \mathrm{C}(X, \mathbf{R})$ and wish to show that $f + g \in I_{\mathfrak{m}}$ and $fh \in I_{\mathfrak{m}}$. The former claim follows from the

---

[7] We denote by $\mathrm{Sub}(X)$ the set of all subsets of X.

observation $\mathrm{V}_X(f+g) \supset \mathrm{V}_X(f) \cap \mathrm{V}_X(g)$ and Lemma A.15, while the latter claim follows from the observation $\mathrm{V}_X(fh) \supset \mathrm{V}_X(f)$ and Lemma A.15.

Now Equation (A.19) implies that, for the purpose of showing that $\mathfrak{m}$ is a zs-ideal, it suffices to show that $\mathfrak{m} = I_{\mathfrak{m}}$. Clearly, we have $\mathfrak{m} \subset I_{\mathfrak{m}}$. Therefore, the fact that $\mathfrak{m}$ is a maximal ideal implies that, in order to show that $\mathfrak{m} = I_{\mathfrak{m}}$, it suffices to show that $1 \notin I_{\mathfrak{m}}$. This is equivalent to showing that $\varnothing \notin \mathrm{V}_X[\mathfrak{m}]$. For this note that any $f \in \mathfrak{m}$ is not invertible, therefore $\varnothing \neq \mathrm{V}_X(f)$. This finishes the proof. □

A.20 Lemma. *Let $I, J \subset \mathrm{C}(X, \mathbf{R})$ be two zs-ideals. Then I is a radical ideal and $I \cap J$ is a zs-ideal.*

Proof. We start with the first claim. Suppose $f \in \mathrm{rad}(I)$, so $f^n \in I$ for some n. Then we note that $\mathrm{V}_X(f) = \mathrm{V}_X(f^n)$. So the definition of a zs-ideal implies that $f \in I$. In other words, I is radical.

Now we deal with the second claim. We first claim that $\mathrm{V}_X[I \cap J] = \mathrm{V}_X[I] \cap \mathrm{V}_X[J]$. We always have an inclusion $\mathrm{V}_X[I \cap J] \subset \mathrm{V}_X[I] \cap \mathrm{V}_X[J]$, so it suffices to show that $\mathrm{V}_X[I] \cap \mathrm{V}_X[J] \subset \mathrm{V}_X[I \cap J]$. Pick $Z \in \mathrm{V}_X[I] \cap \mathrm{V}_X[J]$. By definition, this means that there are elements $f \in I$ and $g \in J$ such that $Z = \mathrm{V}_X(f) = \mathrm{V}_X(g)$. Since J is a zs-ideal, it implies that $f \in J$. Therefore, $f \in I \cap J$ and, hence, $Z \in \mathrm{V}_X[I \cap J]$.

Now let $f \in \mathrm{C}(X, \mathbf{R})$ be a continuous function such that $\mathrm{V}_X(f) \in \mathrm{V}_X[I \cap J] = \mathrm{V}_X[I] \cap \mathrm{V}_X[J]$. Then we use the fact that both I and J are zs-ideals to conclude that $f \in I \cap J$, i.e., $I \cap J$ is a zs-ideal. □

A.21 Remark. Lemma A.20 implies that the ideal $(x) \in \mathrm{C}(\mathbf{R}, \mathbf{R})$ is *not* a zs-ideal.

A.22 Lemma [27, Theorem 2.9]**.** *Let $I \subset \mathrm{C}(X, \mathbf{R})$ be a zs-ideal. Then the following are equivalent:*

(1) *The ideal I is prime;*

(2) *The ideal I contains a prime ideal;*

(3) *For any $f, g \in \mathrm{C}(X, \mathbf{R})$ such that $fg = 0$, we have $f \in I$ or $g \in I$;*

(4) *For every $f \in \mathrm{C}(X, \mathbf{R})$, there is a subset $Z \subset X$ such that $Z \in \mathrm{V}_X[I]$ and $f|_Z$ does not change its sign.*

Proof. The implications (1)⇒(2) and (2)⇒(3) are trivial.

Now we show (3)⇒(4). We start by considering the continuous functions $f^+ \coloneqq \max(f, 0)$ and $f^- \coloneqq \min(f, 0)$. Then clearly we have

$$f^+ \cdot f^- = 0\,,$$

so we have $f^+ \in I$ or $f^- \in I$. Suppose $f^+ \in I$ (the other case is similar), then

$$\{x \in X \mid f(x) \leq 0\} = \mathrm{V}_X(f^+) \in \mathrm{V}_X[I]\,.$$

Now we show (4)⇒(1). We pick two continuous functions $f, g \in \mathrm{C}(X, \mathbf{R})$ such that $fg \in I$ and wish to show that $f \in I$ or $g \in I$. For this, we consider the continuous function $h = |f| - |g|$. Our assumption implies that there is a zero set $Z \in \mathrm{V}_X[I]$ such that $h|_Z$ is, say, nonnegative (the other case is similar). Note that if $f(x) = 0$ and $x \in Z$, then $h(x) = -|g(x)| \geq 0$. Hence, $h(x) = g(x) = 0$ for such $x \in X$. So we conclude that $Z \cap \mathrm{V}_X(fg) = Z \cap (\mathrm{V}_X(f) \cup \mathrm{V}_X(g)) = Z \cap \mathrm{V}_X(g)$. Therefore, we see that $\mathrm{V}_X(g) \in \mathrm{V}_X[I]$ by virtue of Lemma A.15 and the following sequence of inclusions:

$$\mathrm{V}_X(g) \supset Z \cap \mathrm{V}_X(g) = Z \cap \mathrm{V}_X(fg)$$

Therefore, we conclude that $g \in I$ since I is a zs-ideal. □

We are almost ready to show that $\mathrm{C}(X, \mathbf{R})$ is a pm-ring. We only need the following abstract commutative algebra lemma.

A.23 Lemma. *Let R be a ring and let $\mathfrak{p}_1, \mathfrak{p}_2 \subset R$ be prime ideals such that neither of them is contained in the other. Then $\mathfrak{p}_1 \cap \mathfrak{p}_2$ is not a prime ideal.*

Proof. Choose $t \in \mathfrak{p}_1 \smallsetminus \mathfrak{p}_2$ and $s \in \mathfrak{p}_2 \smallsetminus \mathfrak{p}_1$. Then $st \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ but $s \notin \mathfrak{p}_1 \cap \mathfrak{p}_2$ and $t \notin \mathfrak{p}_1 \cap \mathfrak{p}_2$. □

A.24 Theorem [27, Theorem 2.11]**.** *The ring* $\mathrm{C}(X, \mathbf{R})$ *is a pm-ring.*

Proof. Every prime ideal $\mathfrak{p} \subset \mathrm{C}(X, \mathbf{R})$ is contained in some maximal ideal, so it suffices to show that $\mathfrak{p}$ cannot be contained in two different maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$. We set $I \coloneqq \mathfrak{m}_1 \cap \mathfrak{m}_2$. Then Theorem A.18 and Lemma A.20 imply that I is a zs-ideal. By construction, we have an inclusion $\mathfrak{p} \subset I$. Therefore, Lemma A.22 ensures that I is a prime ideal. However, this contradicts Lemma A.23. Hence, there is only one maximal ideal containing $\mathfrak{p}$. □

A.25 Corollary. *Let X be a topological space. Then* $\mathrm{MSpec}\big(\mathrm{C}(X, \mathbf{R})\big)$ *is a compact Hausdorff topological space.*

Proof. This follows directly from Theorem A.24 and Corollary A.10. □

We now address the fact that $\mathrm{C}(X, \mathbf{C})$ is a pm-ring.

A.26 Lemma. *The natural map* $\mathrm{C}(X, \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C} \to \mathrm{C}(X, \mathbf{C})$ *is an isomorphism.*

Proof. First, we note that the question is equivalent to showing that the natural map $\mathrm{C}(X, \mathbf{R}) \oplus i \cdot \mathrm{C}(X, \mathbf{R}) \to \mathrm{C}(X, \mathbf{C})$ is an isomorphism. In other words, we need to show that any continuous function $f \in \mathrm{C}(X, \mathbf{C})$ can be uniquely written as $f = g + i \cdot h$ with $g, h \in \mathrm{C}(X, \mathbf{R})$. Uniqueness is clear. To see existence, we note that $f = \mathrm{Re}(f) + i \cdot \mathrm{Im}(f)$. □

A.27 Lemma. *The natural map* $\mathrm{Spec}(\mathrm{C}(X, \mathbf{C})) \to \mathrm{Spec}(\mathrm{C}(X, \mathbf{R}))$ *restricts to a bijection*

$$\mathrm{MSpec}(\mathrm{C}(X, \mathbf{C})) \to \mathrm{MSpec}(\mathrm{C}(X, \mathbf{R})).$$

Proof. By Lemma A.26, $\mathrm{C}(X, \mathbf{R}) \to \mathrm{C}(X, \mathbf{C})$ is a finite ring extension and thus $\mathrm{Spec}(\mathrm{C}(X, \mathbf{C})) \to \mathrm{Spec}(\mathrm{C}(X, \mathbf{R}))$ maps closed points to closed points. To show that it restricts to a bijection on closed points, it suffices to see that for every maximal ideal $\mathfrak{m} \subset \mathrm{C}(X, \mathbf{R})$ with residue field $k_{\mathfrak{m}}$, the tensor product $k_{\mathfrak{m}} \otimes_{\mathrm{C}(X, \mathbf{R})} \mathrm{C}(X, \mathbf{C})$ is a field. By Lemma A.26, this is equivalent to showing that $k_{\mathfrak{m}} \otimes_{\mathbf{R}} \mathbf{C}$ is a field. For this it suffices to show that the equation $X^2 + 1 = 0$ has no solutions in $k_{\mathfrak{m}}$. In other words, we need to show that there are no continuous functions $f \in \mathrm{C}(X, \mathbf{R})$ and $g \in \mathfrak{m}$ such that $f^2 = -1 + g$. Suppose that such functions exist. Then we note that g is not an invertible function since it lies in a maximal ideal. Therefore, there is a point $x \in X$ such that $g(x) = 0$. Thus, we see that $f(x)^2 = -1 + g(x) = -1$. Contradiction, so no such functions exist. □

A.28 Corollary. *The ring* $\mathrm{C}(X, \mathbf{C})$ *is a pm-ring.*

Proof. Let $\mathfrak{P} \subset \mathrm{C}(X, \mathbf{C})$ be a prime ideal and let $\mathfrak{M} \subset \mathrm{C}(X, \mathbf{C})$ be a maximal ideal containing $\mathfrak{P}$. We put $\mathfrak{p} \coloneqq \mathfrak{P} \cap \mathrm{C}(X, \mathbf{R})$ and we set $\mathfrak{m} \subset \mathrm{C}(X, \mathbf{R})$ to be the unique maximal ideal containing $\mathfrak{p}$. Since $\mathrm{Spec}\big(\mathrm{C}(X, \mathbf{C})\big) \to \mathrm{Spec}\big(\mathrm{C}(X, \mathbf{R})\big)$ is a finite morphism (see Lemma A.26), it sends closed points to closed points. So we conclude that $\mathfrak{M} \cap \mathrm{C}(X, \mathbf{R}) = \mathfrak{m}$. Thus the claim follows from Lemma A.27.

□

A.29 Corollary. *The canonical map* $\operatorname{MSpec}(\mathrm{C}(X, \mathbf{C})) \to \operatorname{MSpec}(\mathrm{C}(X, \mathbf{R}))$ *is a homeomorphism.*

Proof. By Corollaries A.10, A.25, and A.28 source and target are both compact Hausdorff spaces, so the claim follows from Lemma A.27. ☐

A.4 Čech–Stone compactification via algebraic geometry

In this subsection, we show that the topological space $\operatorname{MSpec}(\mathrm{C}(X, \mathbf{R}))$ satisfies the universal property of the Čech–Stone for any topological space X; this gives a new proof of the existence of the Čech–Stone compactification and automatically identifies it with $\operatorname{MSpec}(\mathrm{C}(X, \mathbf{R}))$.

A.30 Definition. The *Čech–Stone compactification* of a topological space X is the pair $(\beta(X), i_X)$ of a compact Hausdorff space $\beta(X)$ and a continuous morphism $i_X : X \to \beta(X)$ such that, for every other compact Hausdorff space Y with a continuous map $f : X \to Y$, there is a unique continuous map $\beta(f) \colon \beta(X) \to Y$ such that $f = \beta(f) \circ i_X$.

A.31 Remark. Clearly, the Čech–Stone compactification of X is unique up to a unique homeomorphism if it exists.

We recall (see Construction A.1) that, for every topological space X, we have the natural morphism $\iota_X : X \to \operatorname{MSpec}(\mathrm{C}(X, \mathbf{R}))$. Our goal is to show that the pair $\big(\operatorname{MSpec}\big(\mathrm{C}(X, \mathbf{R})\big), \iota_X\big)$ satisfies the universal property of the Čech–Stone compactification.

A.32 Theorem. *Let X be a compact Hausdorff space. Then the natural map*

$$\iota_X : X \to \operatorname{MSpec}(\mathrm{C}(X, \mathbf{R})) \cong \operatorname{MSpec}(\mathrm{C}(X, \mathbf{C}))$$

is a homeomorphism.

Proof. The homeomorphism $\operatorname{MSpec}(\mathrm{C}(X, \mathbf{R})) \cong \operatorname{MSpec}(\mathrm{C}(X, \mathbf{C}))$ is simply Corollary A.29. Thus we only prove that ι_X is an isomorphism.

Step 1: ι_X is injective. To show injectivity of ι_X, it suffices to show that any two points $x, y \in X$ can be separated by a continuous function $f : X \to \mathbf{R}$. More precisely, we need to find a continuous function $f : X \to \mathbf{R}$ such that $f(x) = 0$ and $f(y) \neq 0$. Such a function exists by Urysohn's Lemma [64, Theorem 33.1].

Step 2: ι_X has dense image. This follows directly from Lemma A.3.

Step 3: ι_X is a homeomorphism. Since X is quasi-compact, we conclude that its image $\iota_X(X)$ is also quasi-compact. Since $\operatorname{MSpec}(\mathrm{C}(X, \mathbf{R}))$ is Hausdorff (see Corollary A.25), we conclude that $\iota_X(X)$ is closed. Since $\iota_X(X) \subset \operatorname{MSpec}(\mathrm{C}(X, \mathbf{R}))$, we conclude that ι_X must be surjective. Therefore, ι_X is a bijective continuous map between compact Hausdorff spaces (see Corollary A.25), so it is a homeomorphism by virtue of [STK, Tag 08YE]. ☐

A.33 Lemma. *Let $f : X \to Y$ be a continuous map of topological spaces. Then there is a unique continuous map $\widetilde{f} \colon \operatorname{MSpec}\big(\mathrm{C}(X, \mathbf{R})\big) \to \operatorname{MSpec}\big(\mathrm{C}(Y, \mathbf{R})\big)$ that makes the square*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\iota_X"'] & Y \arrow[d, "\iota_Y"] \\
\operatorname{MSpec}\big(\mathrm{C}(X, \mathbf{R})\big) \arrow[r, dotted, "\widetilde{f}"'] & \operatorname{MSpec}\big(\mathrm{C}(Y, \mathbf{R})\big)
\end{tikzcd}$$

commute.

Proof. First, we note that $\iota_X(X) \subset \mathrm{MSpec}\big(\mathrm{C}(X, \mathbf{R})\big)$ is dense by Lemma A.3. Therefore, $\widetilde{f}$ is unique if exists. For the existence, we denote by $f^* : \mathrm{C}(Y, \mathbf{R}) \to \mathrm{C}(X, \mathbf{R})$ the natural pullback homomorphism. Then $\widetilde{f} = \mathrm{MSpec}(f^*)$ does the job (see Theorem A.24 and Definition A.11). ☐

A.34 Theorem. *Let X be a topological space, let Y be a compact Hausdorff space, and let $f : X \to Y$ be a continuous map. Then there is a unique continuous map $\widetilde{f} : \mathrm{MSpec}\big(\mathrm{C}(X, \mathbf{R})\big) \to Y$ that makes the triangle*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\iota_X"'] & Y \\
\mathrm{MSpec}\big(\mathrm{C}(X, \mathbf{R})\big) \arrow[ur, dotted, "\widetilde{f}"'] &
\end{tikzcd}$$

commute.

Proof. This follows immediately from Lemma A.33 and Theorem A.32. ☐

A.35 Corollary. *Let X be a topological space. Then the Čech–Stone compactification $(\beta(X), i_X)$ of X exists and $\beta(X) \simeq \mathrm{MSpec}\big(\mathrm{C}(X, \mathbf{R})\big)$.*

Proof. This follows immediately from Theorem A.34 and Remark A.31. ☐

B Galois groups of function fields

It is well-known that there is an isomorphism of profinite groups

$$\widehat{\mathrm{Fr}}_{\mathbf{C}} \simeq \mathrm{Gal}_{\mathbf{C}(T)}$$

between the free profinite group on the underlying set of $\mathbf{C}$ and the absolute Galois group of the function field $\mathbf{C}(T)$. See [19; 38]. Moreover, it seems to be folklore that this isomorphism can be chosen so that the free profinite group generated by an element $a \in \mathbf{C}$ corresponds to a decomposition group of the prime $(T - a)$. See [46, §1.8]. The purpose of this appendix is to record a proof of this folklore statement. Implicitly this is also shown in [50] and we do not claim originality of any of the results in this appendix.

B.1 Notation. Throughout this section we fix an algebraic closure K of the function field $\mathbf{C}(T)$. We write $\mathrm{Gal}_{\mathbf{C}(T)} \coloneqq \mathrm{Gal}(K/\mathbf{C}(T))$.

B.2 Recollection. Write $\overline{\mathbf{C}[T]} \subset K$ for the integral closure of $\mathbf{C}[T]$ in K. For any $a \in \mathbf{C}$ a choice of prime ideal $\bar{a}$ in $\overline{\mathbf{C}[T]}$ lying over $(T-a)$ then determines a decomposition group $\mathrm{D}_{\bar{a}} \subset \mathrm{Gal}_{\mathbf{C}(T)}$. Moreover, if $\bar{a}'$ is another choice of prime above $(T - a)$, then $\mathrm{D}_{\bar{a}'}$ is conjugate to $\mathrm{D}_{\bar{a}}$.

Our goal is to prove the following result, which is a slight refinement of [19, Theorem 2] for $C = \mathbf{C}$.

B.3 Theorem. *There is an isomorphism of profinite groups*

$$\widehat{\mathrm{Fr}}_{\mathbf{C}} \to \mathrm{Gal}_{\mathbf{C}(t)}$$

such that for each $a \in \mathbf{C}$ the image of $\widehat{\mathbf{Z}}(a)$ under this isomorphism is the decomposition group $\mathrm{D}_{\bar{a}|a}$ of a prime $\bar{a}$ lying over $(T - a)$.

B.4 Definition. Let M be a set. Write Σ for the system of finite subsets $S \subset M$ partially ordered by inclusion. Let $((G_S)_{S\in\Sigma}, (\rho_S^T)_{S\subset T})$ be an inverse system of profinite groups with limit $G_M := \lim_{S\in\Sigma} G_S$ and write $\rho_S^M : G_M \to G_S$ for the canonical projection. Let N be either the whole of M, or an element of Σ.

(1) We say that a function $\varphi : N \to G_N$ is *adapted* if $\rho_S^N(\varphi(n)) = 1$ for all finite subsets $S \subset N$ and all $n \notin S$.

(2) We say that a function $\varphi : N \to G_N$ is an *adapted basis* if φ is adapted and if the map $\widehat{\mathrm{Fr}}_N \to G_N$ induced by φ is an isomorphism.

(3) We say that a system $\mathcal{B} = (\mathcal{B}_S)_{S\in\Sigma}$ of sets of functions $\mathcal{B}_S \subset \mathrm{Hom}(S, G_S)$ is a *system of adapted bases* if the following conditions hold.

   (a) For each $S \in \Sigma$, $\mathcal{B}_S \subset \mathrm{Hom}(S, G_S) = \prod_S G_S$ is a nonempty closed subset consisting of adapted bases.

   (b) For each $S \subset T \in \Sigma$, and each $\varphi \in \mathcal{B}_T$, the restriction $S \subset T \xrightarrow{\varphi} G_T \xrightarrow{\rho_S^T} G_S$ is an element of $\mathcal{B}_S$.

B.5 Proposition. *Let M be a set. Write Σ for the poset of finite subsets $S \subset M$ partially ordered by inclusion. Let $((G_S)_{S\in\Sigma}, (\rho_S^T)_{S\subset T})$ be an inverse system of profinite groups with limit $G_M := \lim_{S\in\Sigma} G_S$ and write $\rho_S^M : G_M \to G_S$ for the canonical projection. Let $\mathcal{B}$ be a system of adapted bases. If all the transition maps $\rho_S^T : G_T \to G_S$ are surjective, then there exists an adapted basis $M \to G_M$ such that for each $S \in \Sigma$, the restriction*

$$S \subset M \to G_M \xrightarrow{\rho_S^M} G_S$$

is a basis contained in $\mathcal{B}_S$.

Proof. In [19, Theorem 1], Douady proved the above claim in the case where $\mathcal{B}$ is the system of adapted bases consisting of $\mathcal{B}_S$ the set of all adapted bases $S \to G_S$. However, the argument he gives actually only uses the axiomatic of a general system of adapted bases in the above sense. ☐

We will use the following lemma:

B.6 Lemma. *Let G be a profinite group and let $H, H' \subset G$ be closed subgroups. Let $\alpha : G \to G'$ be a homomorphism of profinite groups. Let*

$$M := \{ g \in G \mid \alpha(g^{-1})\alpha(H)\alpha(g) = \alpha(H') \} .$$

Then M is closed in G.

Proof. We first consider the set

$$M' := \{ g \in G \mid \alpha(g^{-1})\alpha(H)\alpha(g) \subset \alpha(H') \} .$$

For $h \in H$, write

$$M'_h := \{ g \in G \mid \alpha(g^{-1}hg) \in \alpha(H') \} .$$

This is preimage of $\alpha(H') \subset G'$ under the continuous map $G \to G'$ that sends g to $\alpha(g^{-1}hg)$. Since $\alpha(H') \subset G'$ is closed it follows that M'_h is closed. Since

$$M' = \bigcap_{h \in H} M'_h$$

it follows that M' is closed. Now note that the same argument shows that

$$M'' \coloneqq \{ g \in G \mid \alpha(g)\alpha(H')\alpha(g)^{-1} \subset \alpha(H) \}$$

is closed. Thus $M = M' \cap M''$ is closed. □

Proof of Theorem B.3. Our choice of algebraic closure yields an isomorphism

$$\mathrm{Gal}_{\mathbf{C}(T)} \xrightarrow{\sim} \lim_{S \subset \mathbf{C} \text{ finite}} \pi_1^{\text{ét}}(\mathbf{A}^1 \smallsetminus S, \bar{\eta}) .$$

Let us write $G_S = \pi_1^{\text{ét}}(\mathbf{A}^1 \smallsetminus S, \bar{\eta})$. We want to apply Proposition B.5 to this inverse systems of groups and the system of adapted bases $\mathcal{B}_S$ that consists of those maps $\varphi \colon S \to G_S$ that are adapted bases and for any $s \in S$, the subgroup $\widehat{\mathbf{Z}}(\varphi(s))$ is (conjugate to) a decomposition group at s. To see that $(\mathcal{B}_S)_S$ is a system of adapted bases, we need to show that the conditions Definition B.4-(3.a) and Definition B.4-(3.b) are satisfied. It is clear that (3.b) is satisfied, so we only check (3.a). We start by verifying that $\mathcal{B}_S \subset \mathrm{Hom}(S, G_S)$ is closed. To this end, note that the larger subset $\mathcal{B}_S^{\text{all}} \subset \mathrm{Hom}(S, G_S)$, consisting of all adapted bases is closed, see the beginning of the proof of [75, Proposition 3.4.9]. To conclude, it suffices to see that for all $s \in S$ the subset $\Sigma_s \subset G_S$, consisting of those $\sigma \in G_S$ with the property that $\widehat{\mathbf{Z}}(\sigma)$ is a decomposition group at s, is closed. Indeed, in this case

$$\mathcal{B}_S = \mathcal{B}_S^{\text{all}} \cap \prod_{s \in S} \Sigma_s \subset \mathrm{Hom}(S, G_S) = \prod_S G_S .$$

is seen to be an intersection of closed subsets, hence itself closed. Fix one decomposition group D_s at s. Since $\mathrm{D}_s \simeq \widehat{\mathbf{Z}}$, the subset $N \subset \mathrm{D}_s$ of elements that topologically generate D_s is closed. Now observe that Σ_s agrees with the image of the continuous map

$$N \times G_S \to G_S; \quad (n, g) \mapsto g^{-1}ng$$

and is therefore closed, since the domain is compact. Finally, we need to check that $\mathcal{B}_S \neq \emptyset$. Choose a point $x \in \mathbf{C} \smallsetminus S$ and an étale path $\alpha \colon \bar{\eta} \rightsquigarrow x$ and consider the isomorphism

$$\psi \colon \pi_1^{\text{top}}(\mathbf{C} \smallsetminus S, x)^\wedge \xrightarrow{\sim} \pi_1^{\text{ét}}(\mathbf{A}^1_{\mathbf{C}} \smallsetminus S, x) \xrightarrow{\sim} \pi_1^{\text{ét}}(\mathbf{A}^1_{\mathbf{C}} \smallsetminus S, \bar{\eta})$$

obtained from the Riemann existence theorem and conjugation with α^{-1}. Recall that $\pi_1^{\text{top}}(\mathbf{C} \smallsetminus S, x)$ is freely generated by simple loops γ_s at x around s, that do not loop around other points in S. Then $(s \mapsto \psi(\gamma_s))$ is clearly an adapted basis and furthermore $\psi(\gamma_s)$ generates a decomposition group at s. Thus $(s \mapsto \psi(\gamma_s)) \in \mathcal{B}_S$.

By applying Proposition B.5, we obtain an isomorphism $\varphi \colon \widehat{\mathrm{Fr}}_{\mathbf{C}} \xrightarrow{\sim} \mathrm{Gal}_{\mathbf{C}(T)}$ with the property that for all finite subsets $S \subset \mathbf{C}$ and $a \in S$, $(\rho_S^{\mathbf{C}} \circ \varphi)(a)$ generates a decomposition group at a in G_S. We now show that $\varphi(a)$ generates a decomposition group at a in $\mathrm{Gal}_{\mathbf{C}(T)}$ for any $a \in \mathbf{C}$. To this end, fix one decomposition group $\mathrm{D}_a \subset \mathrm{Gal}_{\mathbf{C}(T)}$ of a. By the above, for every finite subset $S \subset \mathbf{C}$ there exists some $g \in \mathrm{Gal}_{\mathbf{C}(T)}$ such that $\widehat{\mathbf{Z}}(\varphi(a)) = g^{-1}\mathrm{D}_a g$ in G_S. Now by Lemma B.6 the set C_S of all such g is closed. Therefore $\bigcap_S C_S = \lim_S C_S$ is nonempty as a cofiltered limit of nonempty compact Hausdorff spaces. By construction, any element $g \in \bigcap_S C_S$ has the property that $\widehat{\mathbf{Z}}(\varphi(a)) = g^{-1}\mathrm{D}_a g$ holds after projecting to G_S simultaneously for all $S \subset \mathbf{C}$ finite. Since both D_a and $\widehat{\mathbf{Z}}(\varphi(a))$ are closed subgroups of $\mathrm{Gal}_{\mathbf{C}(T)} = \lim_{S \subset \mathbf{C} \text{ finite}} G_S$, this shows that indeed $g^{-1}\mathrm{D}_a g = \widehat{\mathbf{Z}}(\varphi(a))$. In particular, $\varphi(a)$ generates a decomposition group as desired. □

C A profinite analogue of Quillen's Theorem B

The goal of this appendix is to prove Theorem C.7, an analogue of Quillen's Theorem B after completion at a set of primes. Most of the material here is a part of the sixth author's thesis [81, §7.3]. Nevertheless, here the main result is formulated slightly more generally and the exposition was changed to make it more readable for those less familiar with the theory of internal higher categories developed by the fifth and sixth authors.

C.1 Quillen's Theorem B

Given a functor of ∞-categories $f\colon \mathcal{C} \to \mathcal{D}$, Quillen's Theorem B [65, Theorem B] gives a way of calculating the homotopy fiber of the induced map of classifying anima $\mathrm{B}f\colon \mathrm{B}\mathcal{C} \to \mathrm{B}\mathcal{D}$. We begin this appendix by giving a short and model-independent proof of Theorem B that is easier to generalize than Quillen's original argument.

C.1 Theorem (Quillen's Theorem B)**.** *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories such that for any $d \to d' \in \mathcal{D}$ the induced map*

$$\mathrm{B}\mathcal{C}_{/d} \to \mathrm{B}\mathcal{C}_{/d'}$$

is an equivalence. Then for any $d \in \mathcal{D}$, the commutative diagram

$$\begin{tikzcd}
\mathrm{B}\mathcal{C}_{/d} \arrow[r] \arrow[d] & \mathrm{B}\mathcal{C} \arrow[d, "\mathrm{B}f"] \\
* \simeq \mathrm{B}\mathcal{D}_{/d} \arrow[r] & \mathrm{B}\mathcal{D}
\end{tikzcd}$$

is a cartesian square of anima.

The proof rests on the following observation:

C.2 Proposition. *Let $p\colon \mathcal{F} \to \mathcal{D}$ be a left fibration with corresponding straightened functor $\tilde{p}\colon \mathcal{D} \to$ **Ani**. If for each map $s\colon d \to d'$ in $\mathcal{D}$, the induced map $\tilde{p}(s)$ is an equivalence, then for each $d \in \mathcal{D}$, the square*

$$\begin{tikzcd}
\mathcal{F}_d \arrow[r] \arrow[d] & \mathrm{B}\mathcal{F} \arrow[d, "\mathrm{B}p"] \\
* \arrow[r, "d"'] & \mathrm{B}\mathcal{D}
\end{tikzcd}$$

is cartesian.

Proof. By assumption, $\tilde{p}\colon \mathcal{D} \to$ **Ani** factors through the unit map $\mathcal{D} \to \mathrm{B}\mathcal{D}$. Pulling back the universal left fibration, we thus get a diagram

$$\begin{tikzcd}
\mathcal{F}_d \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F} \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Ani}_{*/} \arrow[d] \\
* \arrow[r, "d"'] & \mathcal{D} \arrow[r] \arrow[rr, bend right, "\tilde{p}"'] & \mathrm{B}\mathcal{D} \arrow[r] & \mathbf{Ani}
\end{tikzcd}$$

in which all squares are cartesian. Note that since left fibrations are conservative and $\mathrm{B}\mathcal{D}$ is an anima, $\mathcal{F}'$ is an anima. Since $\mathrm{B}\colon \mathbf{Cat}_\infty \to \mathbf{Ani}$ is locally cartesian (see (5.3)), by applying B to the middle and left-hand squares, we get another diagram

$$\begin{tikzcd}
\mathcal{F}_d \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{B}\mathcal{F} \arrow[r, "\sim"] \arrow[d, "\mathrm{B}p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F}' \arrow[d] \\
* \arrow[r, "d"'] & \mathrm{B}\mathcal{D} \arrow[r, "\mathrm{id}"'] & \mathrm{B}\mathcal{D}
\end{tikzcd}$$

in which all squares are cartesian, completing the proof. □

C.3 Remark. The assumptions of Proposition C.2 are satisfied whenever the left fibration p is additionally a right fibration, i.e., a Kan fibration.

We now need to build the correct left fibration to which we can apply Proposition C.2. For this we need the following.

C.4 Notation. Let $\mathcal{D}$ be an ∞-category. We write $\mathrm{Cocart}(\mathcal{D}) \subset \mathbf{Cat}_{\infty,/\mathcal{D}}$ for the subcategory with objects cocartesian fibrations $p\colon \mathcal{F} \to \mathcal{D}$ and morphisms the cocartesian functors. We write

$$\mathrm{LFib}(\mathcal{D}) \subset \mathrm{Cocart}(\mathcal{D})$$

for the full subcategory spanned by the left fibrations. Note that $\mathrm{LFib}(\mathcal{D})$ is also a full subcategory of $\mathbf{Cat}_{\infty,/\mathcal{D}}$.

C.5 Recollection. For an ∞-category $\mathcal{D}$, the inclusion $\mathrm{Fun}(\mathcal{D}, \mathbf{Ani}) \hookrightarrow \mathrm{Fun}(\mathcal{D}, \mathbf{Cat}_\infty)$ admits a left adjoint given by postcomposition with $\mathrm{B}\colon \mathbf{Cat}_\infty \to \mathbf{Ani}$. Under the straightening-unstraightening equivalence, this corresponds to a left adjoint of the inclusion

$$\mathrm{LFib}(\mathcal{D}) \hookrightarrow \mathrm{Cocart}(\mathcal{D})\,.$$

Explicitly, this adjoint sends a cocartesian fibration $p\colon \mathcal{P} \to \mathcal{D}$ to the unique left fibration $L(p)\colon \mathcal{F} \to \mathcal{D}$ that fits in a commutative triangle

$$\begin{tikzcd}
\mathcal{P} \arrow[rr, "\iota"] \arrow[dr, "p"'] & & \mathcal{F} \arrow[dl, "L(p)"] \\
& \mathcal{D} &
\end{tikzcd},$$

where the functor ι is initial. Indeed, such a factorization exists because left fibrations are the right class in the initial-left fibration factorization system, see, e.g., [57, § 4.1]. This also implies that for any left fibration $q\colon \mathcal{G} \to \mathcal{D}$, there is a natural equivalence

$$\mathrm{Map}_{\mathrm{Cocart}(\mathcal{D})}(p,q) \simeq \mathrm{Map}_{\mathbf{Cat}_{\infty,/\mathcal{D}}}(p,q) \simeq \mathrm{Map}_{\mathrm{LFib}(\mathcal{D})}(L(p),q)\,.$$

Here, left-hand equivalence holds since for left fibrations every edge is cocartesian. The right-hand equivalence follows from the fact that the left fibrations are the right class of a factorization system [HTT, Lemma 5.2.8.19].

In order to prove Theorem C.1, we fix some notation regarding oriented fiber products of ∞-categories.

C.6 Recollection. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We consider the oriented fiber product (also called comma ∞-category) $\mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D}$ defined via the pullback

$$\begin{tikzcd}
\mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], \mathcal{D}) \arrow[d, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"] \\
\mathcal{C} \times \mathcal{D} \arrow[r, "f \times \mathrm{id}_{\mathcal{D}}"'] & \mathcal{D} \times \mathcal{D}
\end{tikzcd}$$

in $\mathbf{Cat}_\infty$. Note that by the universal property of the pullback, the functors $(\mathrm{id}_{\mathcal{C}}, f)\colon \mathcal{C} \to \mathcal{C} \times \mathcal{D}$ and

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{\mathrm{id}_{(-)}} \mathrm{Fun}([1], \mathcal{D})$$

induce a functor $j\colon \mathcal{C} \to \mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D}$. By [HTT, Corollary 2.4.7.12], the projection $\mathrm{pr}_2\colon \mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D} \to \mathcal{D}$ is a cocartesian fibration. The cocartesian fibration pr_2 classifies the functor

$$\mathcal{D} \to \mathbf{Cat}_\infty\,, \ d \mapsto \mathcal{C}_{/d}\,.$$

Furthermore, f factors as

$$\mathcal{C} \xrightarrow{j} \mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D} \xrightarrow{\mathrm{pr}_2} \mathcal{D}\,,$$

and j admits a right adjoint given by projecting to the first factor.

Proof of Theorem C.1. We apply the left adjoint L of Recollection C.5 to the cocartesian fibration $\mathrm{pr}_2\colon \mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D} \to \mathcal{D}$. Our assumptions precisely say that the resulting left fibration $L(\mathrm{pr}_2)\colon \mathcal{F} \to \mathcal{D}$ satisfies the assumptions of Proposition C.2. Thus we get a commutative diagram

$$\begin{tikzcd}
\mathrm{B}\mathcal{C}_{/d} \arrow[r] \arrow[d] & \mathrm{B}\mathcal{C} \arrow[r, "\mathrm{B}j"] \arrow[d, "\mathrm{B}f"] & \mathrm{B}(\mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D}) \arrow[r, "\mathrm{B}\iota"] & \mathrm{B}\mathcal{F} \arrow[d, "\mathrm{B}L(\mathrm{pr}_2)"] \\
* \simeq \mathrm{B}\mathcal{D}_{/d} \arrow[r, "d"'] & \mathrm{B}\mathcal{D} \arrow[rr, "\mathrm{id}"'] & & \mathrm{B}\mathcal{D}\,,
\end{tikzcd}$$

where the outer square is cartesian. Furthermore, since B inverts adjoints and initial functors (see, e.g., [15, Corollary 2.11(4) & Remark 2.20]), the right square is cartesian. Thus the left square is cartesian, as desired. ☐

C.2 Profinite Theorem B

The goal of this subsection is to prove a variant of Quillen's Theorem B for profinite categories following the general strategy of §C.1. The main ingredient of the proof of Theorem C.1 was the straightening-unstraightening equivalence. However profinite categories are not well-behaved enough to admit a full straightening-unstraightening equivalence. The solution is to embed profinite categories into condensed categories, where we have a straightening-unstraightening equivalence thanks to [58, Theorem 6.3.1]. The precise theorem we aim to prove in this subsection is the following:

C.7 Theorem. *Let Σ be a nonempty set of prime numbers. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a map in* $\mathrm{Cat}(\mathrm{Pro}(\mathbf{Ani}_\pi))$ *such that for any map $d \to d'$ in $\mathcal{D}$ the map of condensed anima*

$$\mathrm{B}^{\mathrm{cond}}(\mathcal{C}_{/d}) \to \mathrm{B}^{\mathrm{cond}}(\mathcal{C}_{/d'})$$

becomes an equivalence after Σ*-completion. Then, for all* $d \in \mathcal{D}$*, the induced map*

$$\mathrm{B}^{\mathrm{cond}}(\mathcal{C}_{/d}) \to \mathrm{fib}_d(\mathrm{B}^{\mathrm{cond}} f)$$

becomes an equivalence after Σ*-completion.*

As mentioned above, straightening-unstraightening plays a crucial role in our proof. Thus, we begin by defining cocartesian fibrations of condensed ∞-categories.

C.8 Definition. Let $\mathcal{C}$ be a condensed ∞-category.

(1) A functor $p\colon \mathcal{P} \to \mathcal{C}$ of condensed ∞-categories is a *cocartesian fibration* if for each $S \in \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$, the induced functor $p(S)\colon \mathcal{P}(S) \to \mathcal{C}(S)$ is a cocartesian fibration and, furthermore, for each map $\alpha\colon T \to S$ in $\mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$, the functor $\alpha^*\colon \mathcal{P}(S) \to \mathcal{P}(T)$ sends $p(S)$-cocartesian morphisms to $p(T)$-cocartesian morphisms.

(2) A cocartesian fibration $p\colon \mathcal{P} \to \mathcal{C}$ is a *left fibration* if for each $S \in \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$, the induced functor $p(S)\colon \mathcal{P}(S) \to \mathcal{C}(S)$ is a left fibration.

(3) We write $\mathrm{Cocart}^{\mathrm{cts}}(\mathcal{C})$ for the subcategory of $\mathrm{Cond}(\mathbf{Cat}_\infty)_{/\mathcal{C}}$ with objects the cocartesian fibrations and morphisms the functors $f\colon \mathcal{P} \to \mathcal{Q}$ over $\mathcal{C}$ such that for every $S \in \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$, the functor $f(S)$ preserves cocartesian morphisms. We write $\mathrm{LFib}^{\mathrm{cts}}(\mathcal{C}) \subset \mathrm{Cocart}^{\mathrm{cts}}(\mathcal{C})$ for the full subcategory spanned by the cocartesian fibrations.

C.9 Remark. Let us denote by $\mathrm{Fun}^{\mathrm{cocart}}([1], \mathbf{Cat}_\infty)$ the subcategory of $\mathrm{Fun}([1], \mathbf{Cat}_\infty)$ with objects cocartesian fibrations and a morphism from $p\colon \mathcal{P} \to \mathcal{C}$ to $p'\colon \mathcal{P}' \to \mathcal{C}'$ is a square squares

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "f"] \arrow[d, "p"'] & \mathcal{P}' \arrow[d, "p'"] \\
\mathcal{C} \arrow[r] & \mathcal{C}'
\end{tikzcd}$$

such that f sends p-cocartesian morphisms to p'-cocartesian morphisms. Then combining [25, Theorem 4.5] and [HA, Proposition 7.3.2.6] shows that the inclusion

$$\mathrm{Fun}^{\mathrm{cocart}}([1], \mathbf{Cat}_\infty) \hookrightarrow \mathrm{Fun}([1], \mathbf{Cat}_\infty)$$

is a right adjoint. In particular, the inclusion preserves limits.

Let $p\colon \mathcal{P} \to \mathcal{C}$ be a functor of condensed ∞-categories. The closure of $\mathrm{Fun}^{\mathrm{cocart}}([1], \mathbf{Cat}_\infty)$ under limits in $\mathrm{Fun}([1], \mathbf{Cat}_\infty)$ shows that if p is a cocartesian fibration, then any map of condensed anima $s\colon B \to A$, the functor s^* in the square

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{cts}}(A, \mathcal{P}) \arrow[r, "s^*"] \arrow[d, "p_*"'] & \mathrm{Fun}^{\mathrm{cts}}(B, \mathcal{P}) \arrow[d, "p_*"] \\
\mathrm{Fun}^{\mathrm{cts}}(A, \mathcal{C}) \arrow[r, "s^*"'] & \mathrm{Fun}^{\mathrm{cts}}(B, \mathcal{C})
\end{tikzcd}$$

sends $p(A)$-cocartesian morphisms to $p(B)$-cocartesian morphisms. Thus, using [58, Proposition 3.17], it follows that our definition of cocartesian fibration agrees with the definition given in [58] in the case $\mathcal{B} = \mathrm{Cond}(\mathbf{Ani})$.

C.10 Remark. By Remark 6.4, a functor of condensed ∞-categories $p : \mathcal{F} \to \mathcal{C}$ is a left fibration in the sense of Definition C.8 if and only if p^{op} is a right fibration in the sense of Definition 6.2. Furthermore, if $\mathcal{F} \to \mathcal{C}$ is a left fibration and $\mathcal{P} \to \mathcal{C}$ is a cocartesian fibration, then every functor $f : \mathcal{P} \to \mathcal{F}$ of condensed ∞-categories over $\mathcal{C}$ is a map in $\mathrm{Cocart}^{\mathrm{cts}}(\mathcal{C})$.

For the condensed version of straighetning-unstraightening, we need to consider the condensed ∞-category of condensed ∞-categories:

C.11 Definition. We write $\mathbf{Cond}(\mathbf{Cat}_\infty)$ for the condensed ∞-category given by the assignment

$$\mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})^{\mathrm{op}} \ni S \mapsto \mathrm{Cat}(\mathrm{Cond}(\mathbf{Ani})_{/S}) .$$

C.12 Theorem ([58, Theorem 6.3.1] and [57, Theorem 4.5.1])**.** *There is an natural equivalence of ∞-categories*

$$\mathrm{Cocart}^{\mathrm{cts}}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{cts}}(\mathcal{C}, \mathbf{Cond}(\mathbf{Cat}_\infty))$$

Moreover, this equivalence restricts to a natural equivalence

$$\mathrm{LFib}^{\mathrm{cts}}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{cts}}(\mathcal{C}, \mathbf{Cond}(\mathbf{Ani})) .$$

We also have the following analogue of Recollection C.5 for condensed ∞-categories:

C.13 Observation. Recall that the inclusion $\mathrm{Cond}(\mathbf{Ani}) \hookrightarrow \mathrm{Cond}(\mathbf{Cat}_\infty)$ admits a left adjoint $\mathrm{B}^{\mathrm{cond}} : \mathrm{Cond}(\mathbf{Cat}_\infty) \to \mathrm{Cond}(\mathbf{Ani})$. It is easy to see that both of these functors are compatible with basechange and therefore lift to an adjunction of condensed ∞-categories

$$\iota : \mathbf{Cond}(\mathbf{Ani}) \leftrightarrows \mathbf{Cond}(\mathbf{Cat}_\infty) : \mathrm{B}^{\mathrm{cond}} ,$$

i.e., an adjunction in the $(\infty, 2)$-category of condensed ∞-categories. See also [59, Definition 3.1.1 and Proposition 3.2.14]. Thus the induced functor

$$\mathrm{Fun}^{\mathrm{cts}}(\mathcal{C}, \mathbf{Cond}(\mathbf{Ani})) \to \mathrm{Fun}^{\mathrm{cts}}(\mathcal{C}, \mathbf{Cond}(\mathbf{Cat}_\infty))$$

admits a left adjoint given by postcomposition with $\mathrm{B}^{\mathrm{cond}}$. Under the straightening-unstraightening equivalence of Theorem C.12, this corresponds to a left adjoint L of the inclusion

$$\mathrm{LFib}^{\mathrm{cts}}(\mathcal{C}) \hookrightarrow \mathrm{Cocart}^{\mathrm{cts}}(\mathcal{C}) .$$

Since left fibrations of condensed categories are the right class in the initial-left fibration factorization systems, as in Recollection C.5, it follows from [HTT, Lemma 5.2.8.19] that the left adjoint is given by factoring $\mathcal{P} \to \mathcal{C}$ into an initial functor followed by a left fibration.

To follow the strategy outlined in § C.1, we need a version of Proposition C.2. Now another complication enters. Unlike in § C.1, the maps $\mathrm{B}^{\mathrm{cond}}(\mathcal{C}_{/d}) \to \mathrm{B}^{\mathrm{cond}}(\mathcal{C}_{/d'})$ are not assumed to be equivalences on the nose, but only after Σ-completion. Thus, we also need an analogue of Proposition C.2 that works up to completion. We prove the following statement, which is a variant of [62, Corollary 5.4]:

C.14 Proposition. *Let $\mathcal{X}$ be an ∞-category with colimits and let $L : \mathrm{Cond}(\mathbf{Ani}) \to \mathcal{X}$ be a colimit-preserving functor. Let $\mathcal{C}$ be a condensed ∞-category and $p : \mathcal{F} \to \mathcal{C}$ a left fibration of condensed ∞-categories corresponding via Theorem C.12 to a functor of condensed ∞-categories $\tilde{p} : \mathcal{C} \to \mathbf{Cond}(\mathbf{Ani})$. Assume that for each profinite set S, the functor*

$$\mathcal{C}(S) \xrightarrow{\tilde{p}(S)} \mathrm{Cond}(\mathbf{Ani})_{/S} \longrightarrow \mathrm{Cond}(\mathbf{Ani}) \xrightarrow{L} \mathcal{X}$$

sends all morphisms to equivalences. Then for every $d\colon S \to \mathcal{C}$*, the induced map*

$$\tilde{p}(d)\colon S \times_{\mathcal{C}} \mathcal{F} \to S \times_{\mathrm{B}^{\mathrm{cond}}\mathcal{C}} \mathrm{B}^{\mathrm{cond}}\mathcal{F}$$

becomes an equivalence after applying L*.*

C.15 Recollection. For the proof of the Proposition C.14, we recall that a functor of condensed ∞-categories $f\colon \mathcal{F} \to \mathcal{C}$ is a *Kan fibration* if it is both a left and right fibration. Equivalently, f is Kan fibration if any of the following equivalent conditions is satisfied:

(1) For any $S \in \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$, the functor $f(S)$ is a Kan fibration.

(2) The functor f is right orthogonal to all maps of the form $S \times \{\varepsilon\} \to S \times [n]$, where $S \in \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$, $n \in \mathbf{N}$, and $\varepsilon \in \{0, n\}$.

Indeed, this follows immediately from Remark 6.4 and [57, Lemma 4.1.2].

Proof of Proposition C.14. We work in the ∞-category

$$\mathrm{Cond}(\mathbf{Ani})_{\mathbf{\Delta}} \coloneqq \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cond}(\mathbf{Ani}))$$

of simplicial objects in Cond(**Ani**). We factor $S \to \mathcal{C}$ as $S \xrightarrow{i} T \xrightarrow{f} \mathcal{C}$ where i is contained in the smallest saturated class in $(\mathrm{Cond}(\mathbf{Ani})_{\mathbf{\Delta}})_{/\mathcal{C}}$ containing all maps of the form

$$\begin{tikzcd}
\{\varepsilon\} \times S \arrow[rr] \arrow[dr] & & {[n] \times S} \arrow[dl] \\
& \mathcal{C} &
\end{tikzcd}$$

where $n \in \mathbf{N}$, $\varepsilon \in \{0, n\}$, and $S \in \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$, and f is right orthogonal to these maps. It follows from Recollection C.15 that f is a Kan fibration. Since Kan fibrations are levelwise Kan fibrations, it follows from Remark C.3 that the natural map

$$\mathrm{B}^{\mathrm{cond}}(S \times_{\mathcal{C}} \mathcal{F}) \to S \times_{\mathrm{B}^{\mathrm{cond}}\mathcal{C}} \mathrm{B}^{\mathrm{cond}}\mathcal{F}$$

is an equivalence Thus it suffices to see that the induced map $S \times_{\mathcal{C}} \mathcal{F} \to T \times_{\mathcal{C}} \mathcal{F}$ becomes an equivalence after applying $L \circ \mathrm{B}^{\mathrm{cond}}$.

We note that, by the universality of colimits in $\mathrm{Cond}(\mathbf{Ani})_{\mathbf{\Delta}}$, the class $\mathcal{M}$ of all maps $s\colon A \to B$ in $(\mathrm{Cond}(\mathbf{Ani})_{\mathbf{\Delta}})_{/\mathcal{C}}$, that have the property that

$$L\,\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}}(A \times_{\mathcal{C}} \mathcal{F}) \to L\,\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}}(B \times_{\mathcal{C}} \mathcal{F})$$

is an equivalence is a saturated class in the sense of [57, Definition 2.5.5]. To see that i is contained in $\mathcal{M}$, it therefore suffices to check this for the maps $\{\varepsilon\} \times S \to [n] \times S$, where $S \in \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$ and $\varepsilon \in \{0, n\}$. Note that since the pulled back functor $([n] \times S) \times_{\mathcal{C}} \mathcal{F} \to [n] \times S$ is again a left fibration and the pullback of a final functor along a left fibration is final [57, Proposition 4.4.7], the induced funtor $(\{n\} \times S) \times_{\mathcal{C}} \mathcal{F} \to ([n] \times S) \times_{\mathcal{C}} \mathcal{F}$ is final. In particular,

$$\mathrm{B}^{\mathrm{cond}}((\{n\} \times S) \times_{\mathcal{C}} \mathcal{F}) \to \mathrm{B}^{\mathrm{cond}}(([n] \times S) \times_{\mathcal{C}} \mathcal{F})$$

is an equivalence, so $\{n\} \times S \to [n] \times S$ is in $\mathcal{M}$. Furthermore, under this equivalence, the induced map

$$(\{0\} \times S) \times_{\mathcal{C}} \mathcal{F} \to \mathrm{B}^{\mathrm{cond}}(([n] \times S) \times_{\mathcal{C}} \mathcal{F})$$

is identified with the map $(\{0\} \times S) \times_{\mathcal{C}} \mathcal{F} \to (\{n\} \times S) \times_{\mathcal{C}} \mathcal{F}$ induced by $0 \to n$ in $[n]$ (see Lemma C.16 and Remark C.17 below). But this map is an L-equivalence by assumption. Therefore, i is contained in $\mathcal{M}$, which completes the proof. ☐

C.16 Lemma. *Let $p\colon \mathcal{F} \to \mathcal{C}$ be a left fibration of condensed ∞-categories and $\tilde{p}\colon \mathcal{C} \to \mathbf{Cond}(\mathbf{Ani})$ the straightened functor. Then for any morphism α in $\mathcal{C}(S)$ for some $S \in \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$, given by $\alpha\colon [1] \times S \to \mathcal{C}$, the map $\tilde{p}(\alpha)$ in $\mathrm{Cond}(\mathbf{Ani})_{/S}$ is given by composing*

$$(\{0\} \times S) \times_{\mathcal{C}} \mathcal{F} \to \mathrm{B}^{\mathrm{cond}}(([1] \times S) \times_{\mathcal{C}} \mathcal{F})$$

with the inverse of the equivalence $(\{1\} \times S) \times_{\mathcal{C}} \mathcal{F} \to \mathrm{B}^{\mathrm{cond}}(([1] \times S) \times_{\mathcal{C}} \mathcal{F})$.

Proof. By pulling back along α we may assume that α is the identity. Also we have an equivalence

$$\mathrm{LFib}^{\mathrm{cts}}([1] \times S) \simeq \mathrm{Fun}^{\mathrm{cts}}([1] \times S, \mathbf{Cond}(\mathbf{Ani})) \simeq \mathrm{Fun}([1], \mathrm{Cond}(\mathbf{Ani})_{/S}) .$$

Now observe that $\tilde{p}(\alpha)$ can be computed as

$$\mathrm{ev}_1(\varepsilon\colon \mathrm{const}\,\mathrm{ev}_0\,\tilde{p} \to \tilde{p})$$

Here ε denotes the counit of the adjunction $\mathrm{const}\colon \mathrm{Cond}(\mathbf{Ani})_{/S} \leftrightarrows \mathrm{Fun}([1], \mathrm{Cond}(\mathbf{Ani})_{/S})\colon \mathrm{ev}_0$. Translating to the fibrational perspective via Theorem C.12, we obtain a rectangle

$$\begin{tikzcd}
\{1\} \times F_{\{0\}} \arrow[r] \arrow[d, "\tilde{p}(\alpha)"'] \arrow[dr, phantom, "\lrcorner", very near start] & F_{\{0\}} \times_{\{0\}\times S} ([1] \times S) \cong [1] \times F_{\{0\}} \arrow[d, "\varepsilon"] \\
F_{\{1\}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d] \\
\{1\} \times S \arrow[r] & [1] \times S
\end{tikzcd}$$

and we are done once we see that the composite $F_{\{0\}} \to F_{\{0\}} \times_{\{0\}\times S} ([1] \times S) \to F$ is identified with the inclusion $F_{\{0\}} \to F$ after applying $\mathrm{B}^{\mathrm{cond}}$. But this is clear, since the two inclusions $\{i\} \times F_{\{0\}} \hookrightarrow [1] \times F_{\{0\}}$, $i = 0, 1$, are identified after applying $\mathrm{B}^{\mathrm{cond}}$ and the composite

$$\{0\} \times F_{\{0\}} \hookrightarrow [1] \times F_{\{0\}} \to F$$

yields the inclusion $F_{\{0\}} \to F$ by construction. ☐

C.17 Remark. In the situation of Lemma C.16, we may more generally consider a map $\alpha\colon [n] \times S \to \mathcal{C}$ corresponding to a composable sequence of n arrows in $\mathcal{C}(S)$. Let us denote by $j\colon [1] \to [n]$ the map that sends 0 to 0 and 1 to n. We then get a commutative diagram

$$\begin{tikzcd}
(\{0\} \times S) \times_{\mathcal{C}} \mathcal{F} \arrow[r] \arrow[d, "\mathrm{id}"'] & \mathrm{B}^{\mathrm{cond}}(([1] \times S) \times_{\mathcal{C}} \mathcal{F}) \arrow[d] & (\{1\} \times S) \times_{\mathcal{C}} \mathcal{F} \arrow[l, "\simeq"'] \arrow[d, "\mathrm{id}"] \\
(\{0\} \times S) \times_{\mathcal{C}} \mathcal{F} \arrow[r] & \mathrm{B}^{\mathrm{cond}}(([n] \times S) \times_{\mathcal{C}} \mathcal{F}) & (\{n\} \times S) \times_{\mathcal{C}} \mathcal{F} \arrow[l, "\simeq"]
\end{tikzcd}$$

where the map in the middle is induced by j. Since left fibrations are *smooth* [57, Proposition 4.4.7], the right horizontal maps are equivalences and thus also the vertical map in the middle is an equivalence. It follows that the composite of the lower left map with the inverse of the lower right map is equivalent to $\tilde{p}$ applied to the composite of the n arrows determined by α.

One difference between Proposition C.14 and Theorem C.7 is that in the former we consider fibers over general profinite sets S, while in the latter we only look at fibers over points. To reduce from profinite sets to points, we use the following observation:

C.18 Lemma. *Consider a cartesian square*

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

in $\mathrm{Cond}(\mathbf{Ani})$ *such that* A *is the colimit of a diagram* $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Pro}(\mathbf{Ani}_\pi) \to \mathrm{Cond}(\mathbf{Ani})$ *and* $S, T \in \mathrm{Pro}(\mathbf{Ani}_\Sigma)$. *Then this square remains cartesian after* Σ*-completion.*

Proof. Since $\mathrm{Cond}(\mathbf{Ani})$ is an ∞-topos, geometric realizations are universal in $\mathrm{Cond}(\mathbf{Ani})$. By [33, Example 1.9 and Corollary 1.13], geometric realizations are also universal in $\mathrm{Pro}(\mathbf{Ani}_\Sigma)$. Thus we may assume that $A \in \mathrm{Pro}(\mathbf{Ani}_\pi)$. Since the functor $\mathrm{Pro}(\mathbf{Ani}_\pi) \to \mathrm{Cond}(\mathbf{Ani})$ is fully faithful, the composite

$$\mathrm{Pro}(\mathbf{Ani}_\pi) \longrightarrow \mathrm{Cond}(\mathbf{Ani}) \xrightarrow{(-)^\wedge_\Sigma} \mathrm{Pro}(\mathbf{Ani}_\Sigma)$$

agrees with the Σ-completion functor $(-)^\wedge_\Sigma \colon \mathrm{Pro}(\mathbf{Ani}_\pi) \to \mathrm{Pro}(\mathbf{Ani}_\Sigma)$. The claim now follows from the fact that Σ-completion is locally cartesian [36, Proposition 3.18]. ☐

C.19. Let $f \colon \mathcal{C} \to \mathcal{D}$ be a functor of condensed ∞-categories. We now consider the condensed ∞-category $\mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D}$ defined via the pullback square

$$\begin{tikzcd} \mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}^{\mathrm{cond}}([1], \mathcal{D}) \arrow[d, "{(\mathrm{ev}_0, \mathrm{ev}_1)}"] \\ \mathcal{C} \times \mathcal{D} \arrow[r, "f \times \mathrm{id}_{\mathcal{D}}"'] & \mathcal{D} \times \mathcal{D} \end{tikzcd}$$

as in Recollection C.6. By by [HTT, Corollary 2.4.7.12], the projection $\mathrm{pr}_2 \colon \mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D} \to \mathcal{D}$ is a cocartesian fibration of condensed ∞-categories.

For sake of completeness we verify the following two facts which we have already used for ordinary ∞-categories in the proof of Theorem C.1. First recall that by unstraightening the cocartesian fibration of condensed ∞-categories $\mathrm{ev}_1 \colon \mathrm{Fun}^{\mathrm{cond}}([1], \mathcal{C}) \to \mathcal{C}$, one sees that overcategories of condensed ∞-categories are functorial.

C.20 Proposition. *Let* $f \colon \mathcal{C} \to \mathcal{D}$ *be a functor of condensed* ∞*-categories and consider the natural cocartesian fibration* $\mathrm{pr}_2 \colon \mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D} \to \mathcal{D}$. *Then for every profinite set* S *and morphism* $d \to d'$ *in* $\mathcal{D}(S)$, *the induced functor on fibers is the canonical functor*

$$\mathcal{C}_{/d} = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/d} \longrightarrow \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/d'} = \mathcal{C}_{/d'}$$

in $\mathrm{Cond}(\mathbf{Cat}_\infty)_{/S}$ *induced by the slice functoriality* $\mathcal{D}_{/d} \to \mathcal{D}_{/d'}$.

Proof. We observe that the pullback square

$$\begin{tikzcd} \mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}^{\mathrm{cond}}([1], \mathcal{D}) \arrow[d, "{(\mathrm{ev}_0, \mathrm{ev}_1)}"] \\ \mathcal{C} \times \mathcal{D} \arrow[r, "f \times \mathrm{id}_{\mathcal{D}}"'] & \mathcal{D} \times \mathcal{D} \end{tikzcd}$$

is in fact a pullback square in $\mathrm{Cocart}^{\mathrm{cts}}(\mathcal{D})$. Under the equivalence of Theorem C.12, it therefore corresponds to a cartesian square of functors $\mathcal{D} \to \mathbf{Cond}(\mathbf{Cat}_\infty)$

$$\begin{tikzcd}
\mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D} \arrow[r] \arrow[d] & \mathcal{D}_{/(-)} \arrow[d] \\
\mathrm{const}(\mathcal{C}) \arrow[r, "f"'] & \mathrm{const}(\mathcal{D})
\end{tikzcd}$$

which proves the claim. ☐

C.21 Lemma. *For any functor of condensed ∞-categories $f\colon \mathcal{C} \to \mathcal{D}$, the functor $j\colon \mathcal{C} \to \mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D}$ is a fully faithful left adjoint.*

Proof. The functor j sits inside the commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d, "j"'] & \mathcal{D} \arrow[d, "\mathrm{const}"] \\
\mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D} \arrow[r] \arrow[d] & \mathrm{Fun}^{\mathrm{cond}}([1], \mathcal{D}) \arrow[d, "\mathrm{ev}_0"] \\
\mathcal{C} \arrow[r, "f"'] & \mathcal{D},
\end{tikzcd}$$

in which all squares are cartesian. Since const is the fully faithful left adjoint of ev_0, the proof of [59, Lemma 6.3.9] shows that j is also a fully faithful left adjoint. ☐

Proof of Theorem C.7. We factor f as

$$\mathcal{C} \xrightarrow{j} \mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D} \xrightarrow{\mathrm{pr}_2} \mathcal{D}$$

and apply the left adjoint of Observation C.13 to the cocartesian fibration pr_2. The resulting left fibration $p\colon \mathcal{F} \to \mathcal{C}$ classifies the functor

$$\mathrm{B}^{\mathrm{cond}} \circ \widetilde{\mathrm{pr}}_2 \colon \mathcal{C} \to \mathbf{Cond}(\mathbf{Ani})$$

and is given by factoring

$$\mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D} \xrightarrow{\iota} \mathcal{F} \xrightarrow{p} \mathcal{C}\,,$$

where ι is initial and p is a left fibration. Here, $\widetilde{\mathrm{pr}}_2$ is the unstraightened functor of pr_2.

We now apply Proposition C.14 to the left fibration p, with L the Σ-completion functor

$$(-)^{\wedge}_{\Sigma} \colon \mathrm{Cond}(\mathbf{Ani}) \to \mathrm{Pro}(\mathbf{Ani}_\Sigma)\,.$$

Thus we have to verify that for any $S \in \mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$ and any map $\alpha\colon d \to d' \in \mathcal{C}(S)$, the induced map $\mathrm{B}^{\mathrm{cond}}\widetilde{\mathrm{pr}}_2(\alpha)$ becomes an equivalence after Σ-completion. By construction $\widetilde{\mathrm{pr}}_2(d)$ is defined via a cartesian square

$$\begin{tikzcd}
\widetilde{\mathrm{pr}}_2(d) \arrow[r] \arrow[d] & \mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D} \arrow[d] \\
S \arrow[r, "d"'] & \mathcal{D}
\end{tikzcd}$$

and similarly for $\widetilde{\mathrm{pr}}_2(d')$. It follows that both $\widetilde{\mathrm{pr}}_2(d)$ and $\widetilde{\mathrm{pr}}_2(d')$ are in $\mathrm{Cat}(\mathrm{Pro}(\mathbf{Ani}_\pi))$ since the latter is closed under limits in $\mathrm{Cond}(\mathbf{Cat}_\infty)$. It follows that for any point $s \colon * \to S$ the cartesian square

$$\begin{tikzcd}
\mathrm{B}^{\mathrm{cond}}\widetilde{\mathrm{pr}}_2(d \circ s) \arrow[r] \arrow[d] & \mathrm{B}^{\mathrm{cond}}\widetilde{\mathrm{pr}}_2(d) \arrow[d] \\
* \arrow[r, "s"'] & S
\end{tikzcd}$$

satisfies the assumptions of Lemma C.18, since $\mathrm{B}^{\mathrm{cond}}$ is the geometric realization of the corresponding simplicial object. Thus it remains cartesian after Σ-completion (also the same holds for d' instead of d). By [SAG, Theorem E.3.6.1], equivalences in $\mathrm{Pro}(\mathbf{Ani}_\Sigma)$ can be checked fiberwise. Thus we may thus reduce to the case where $S = *$. But in this case $\mathrm{B}^{\mathrm{cond}}\widetilde{\mathrm{pr}}_2(\alpha)$ is by construction the map

$$\mathrm{B}^{\mathrm{cond}}(\mathcal{C}_{/d}) \to \mathrm{B}^{\mathrm{cond}}(\mathcal{C}_{/d'})\,,$$

which becomes an equivalence after Σ-completion by assumption. Thus, Proposition C.14 shows that in the commutative diagram

$$\begin{tikzcd}
\mathrm{B}^{\mathrm{cond}}\mathcal{C}_{/d} \arrow[r] \arrow[d] & \mathrm{B}^{\mathrm{cond}}\mathcal{C} \arrow[r, "\mathrm{B}^{\mathrm{cond}}j"] \arrow[d, "\mathrm{B}^{\mathrm{cond}}f"] & \mathrm{B}^{\mathrm{cond}}(\mathcal{C} \vec{\times}_{\mathcal{D}} \mathcal{D}) \arrow[r, "\mathrm{B}^{\mathrm{cond}}\iota"] & \mathrm{B}^{\mathrm{cond}}\mathcal{F} \arrow[d, "\mathrm{B}^{\mathrm{cond}}L(\mathrm{pr}_2)"] \\
* \simeq \mathrm{B}^{\mathrm{cond}}\mathcal{D}_{/d} \arrow[r, "d"'] & \mathrm{B}^{\mathrm{cond}}\mathcal{D} \arrow[rr, "\mathrm{id}"'] & & \mathrm{B}^{\mathrm{cond}}\mathcal{D}\,,
\end{tikzcd}$$

the outer square is cartesian. Since $\mathrm{B}^{\mathrm{cond}}$ inverts left adjoints and initial functors of condensed ∞-categories, the claim follows. ☐

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